Review and some open problems in rapidly rotating gases

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What’s new compared to helium?

- At low angular velocities the ground state is a vortex lattice where the cores do not overlap.
- Analogue of H_{c2} in ^{4}He (overlap of vortex cores) is at very high angular velocity, ω, 10^{10} Hz or higher.
- In atom traps this is brought down to the trap frequency, ω_0 ~ 100Hz.
- The downside if the frequency control needed.
- Review – NR Cooper, *Adv Phys*, **57**, 539 (2008)

Magnetic fields and rotation

Uniform magnetic field Hamiltonian

$$\mathcal{H} = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{\mathrm{c}}\mathbf{A}\right)^2$$

Where $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ So that $\nabla \cdot \mathbf{A} = 0$

$$\mathcal{H} = \frac{1}{2m}\left(\mathbf{p}^2 - \frac{e}{\mathrm{c}}(\mathbf{r} \times \mathbf{p}) \cdot \mathbf{B} + \frac{1}{2}\frac{e^2 B^2}{2\mathrm{c}^2}\mathbf{r}^2\right)$$

$$= \frac{1}{2m}\mathbf{p}^2 - (\mathbf{r} \times \mathbf{p}) \cdot \boldsymbol{\Omega}_{\mathrm{c}} + \frac{1}{2}m\boldsymbol{\Omega}_{\mathrm{c}}^{\mathbf{2}}\mathbf{r}^{\mathbf{2}} \,;\, \Omega_{\mathrm{c}} = \frac{eB}{2m\mathrm{c}}$$

Rotating frame

In a rotating frame, a harmonic oscillator has Hamiltonian (with natural frequency ω_0)

$$\mathcal{H} = \frac{1}{2m}\mathbf{p}^2 - \mathbf{L}\cdot\Omega + \frac{1}{2}m\omega_0^2\mathbf{r}^2$$

Note if $\Omega = \omega_0$

Then same Hamiltonian as in the magnetic field case with the cyclotron frequency identified as the angular velocity of the rotating frame.

Instability and Degeneracy

- When the angular velocity of frame equals the natural frequency, the oscillator is on the brink of instability.
- Thus unsurprising that the spectrum of the oscillator changes at that point.

$$E_{n,l} = (2n + |m| + 1)\hbar\omega_0 - m\hbar\Omega$$

- As $\omega_0 - \Omega \to 0$, all positive *m* become degenerate.
- From now on we will work with these Lowest Landau Level (LLL) states, with $n = 0$.

LLL ψ - analytic functions

- The basis set is (here $z = x + iy$)

$$\psi_m(z) = \frac{1}{\sqrt{\pi m!}} z^m \mathrm{e}^{-|z|^2/2}$$

- Hence any wave function in the LLL is an analytic function.
- Degeneracy lifted by either interactions or disorder
- For interactions assume (n is density here)

$$\hbar\omega_0 > gn \text{ and } \hbar(\omega_0 - \Omega) < gn$$

Response of bose liquids to rotation

- Vortex lattice in the case of Helium in a rigid container is due to minimisation of the kinetic energy in the rotating frame. Helium to a good approximation there is incompressible, so only KE matters.
- For a gaseous BEC, different because the interaction energy may be lowered by exploiting the capacity of the cloud to expand. Sheehy and Radzihovsky, *Phys Rev* A **70**, 051602 (2004).
- In LLL, there is nothing but the interactions. (Contrast with top.)

Centrifugal barriers and PE

Wilkin, Gunn and Smith *Phys. Rev. Lett*. **80** 2265 (1998).

- Particles with repulsive interactions can lower their energy by rotating around each other. (I.e. using the centrifugal barrier to ensure there is only small – ideally zero - amplitude for them to experience the potential.
- For contact forces – cold gases – this can be exploited in Laughlin state to render the interactions zero.

$$\psi_{\text{Laugh}} = \prod_{i<j}^{N} (z_i - z_j)^2$$

Pfaffian State

Moore, Read, Nucl. Phys. **B360**, 362 (1991). Wilkin Gunn *Phys Rev Lett* (2000)

States at $L = \frac{1}{2}N(N-2)$ $\quad \psi^{\mathrm{Pf}}(\{z_i\}) = \prod_{i<j}(z_i - z_j)\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)$

Where the Pfaffian is defined as

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) = \mathcal{A} \;\; \frac{1}{(z_1 - z_2)}\frac{1}{(z_3 - z_4)}\cdots\frac{1}{(z_{N-1} - z_N)}\Big]$$

A surprising identity – slabs versus pairing

$$\psi^{\mathrm{Pf}}(\{z_i\}) = \mathcal{S} \prod_{i<j\in\sigma_1}(z_i - z_j)^2 \prod_{k<l\in\sigma_2}(z_k - z_l)^2$$

where the two subsets, σ_1 and σ_2, each have $N/2$ particles ($(N-1)/2$ and $(N+1)/2$ for odd N).

Read-Rezayi states

Read and Rezayi, *Phys Rev B*, **59**, 8084 (1999), Cooper, Wilkin, Gunn *Phys. Rev. Lett*, **87** 120405 (2001).

They are labelled by an integer *k*, where the Pfaffian is *k*=2

$$\psi = S \prod_{i<j}^{N/k} (z_i - z_j)^2 \prod_{l<m=N/k+1}^{2N/k} (z_l - z_m)^2 \cdots \prod_{r<s=(N-1)N/k+1}^{N} (z_r - z_s)^2$$

They are eigenstates of a *k*+1-body generalisation of the repulsive contact interaction. They may occur at “filling fractions” Of $\nu = N^2/2L = k/2$

Correlations - Single Vortex

Wilkin, Gunn and Smith *Phys. Rev. Lett*. **80** 2265 (1998).

At low angular momentum, another way to share the nodes, of a more even nature, occurs:

$$\psi_{L=N}^{\mathrm{exact}} \propto \prod_{i=1}^{N} \left([(z_1 - z_i) + \cdots + (z_n - z_i)] e^{-|z_i|^2/2} \right)$$

$$\psi_{L=N}^{\mathrm{exact}} \propto \prod_{i=1}^{N} \left([z_1 + \cdots + z_n - N z_i] e^{-|z_i|^2/2} \right)$$

Bosons versus fermions

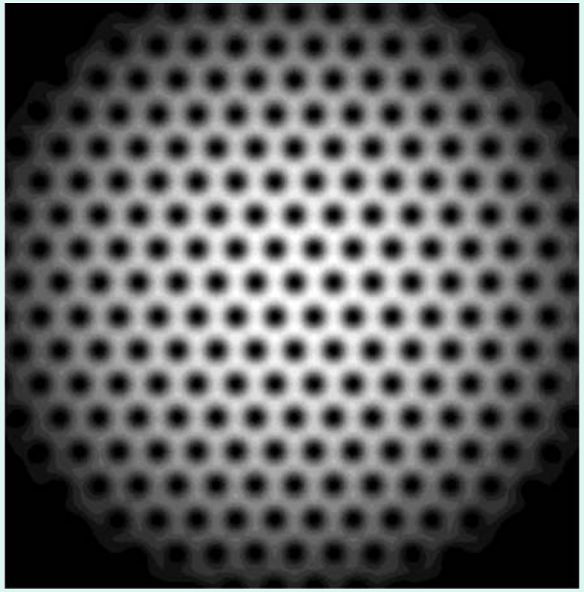
- For bosons $0 \leq \nu \leq \infty$.
- However for fermions, $0 \leq \nu \leq 1$, for a single Landau Level, as the “vortices” must be dense enough (or the angular momentum high enough) to ensure nodes whenever the particles meet.
- For bosons, for $\nu \geq 6$, condensation occurs - the states are uncorrelated in essence. There is no counterpart for fermions.

Mean field LLL

- It has been shown that the mean field solution for the ground state is a vortex lattice (Butts & Rokhsar *Nature* **397** 327 (1999), Ho *Phys Rev Lett* 060403 (2001), Cooper et al *Phys Rev A* **70**, 033604 (2004))

Image from Cooper et al

A proof in the limit that n→∞,U→0, that the mean field theory is exact is in E.H. Lieb and R. Seiringer, *Commun. Math. Phys*. **264** 505 (2006),



Anomalous hydrodynamics

Bourne, Wilkin and Gunn, *Phys Rev Lett*, 96 240401 (2006)

- Before reaching the correlated state, whilst the ground state still resembles a conventional vortex lattice there is a regime where the dynamics is *not* conventional.
- The Bernoulli and continuity equations are drastically modified and there are kinematic relations between gradients of density and phase.

Lowest Landau Level: analyticity

- Anomalous regime occurs when
$$\hbar\omega_0 > gn \text{ and } \hbar(\omega_0 - \omega) < gn$$
- Then the one-particle states are of the form $(z = x + \mathrm{i}y)$
$$\psi(z, z^*, t) = f(z, t)\mathrm{e}^{-|z|^2/2}$$
- *f*(*z*) is analytic, i.e. does not depend on *z**,so
$$\frac{\partial f}{\partial z^*} = 0$$

Density and phase not independent

- As usual can use “hydrodynamic” variables

$$f(z,t) = \rho^{1/2} \mathrm{e}^{\mathrm{i}\phi}$$

- Then $\dfrac{\partial f}{\partial z^*} = 0$ implies that

$$\tfrac{1}{2}\frac{\partial \rho}{\partial x} = \rho\,\frac{\partial \phi}{\partial y} \text{ and } \tfrac{1}{2}\frac{\partial \rho}{\partial y} = -\rho\,\frac{\partial \phi}{\partial x}$$

“Polar” Cauchy-Riemann equations.

“Bernoulli and continuity”

- Equations are highly non-local in terms due to the interparticle interactions – with the other terms just being a trivial advection due to the trap potential.
- Look for a better representation which builds in the analytic structure.

Vortices in the LLL

- Can represent any wavefunction in the LLL by

$$f(z,t) = \sum_{m=0} a_m(t) z^m$$

- Polynomial factorises

$$|\zeta\rangle = f(z,t) = C \prod_{\alpha=1}^{\infty} (z - \zeta_\alpha(t))$$

- And the nodes $\{\zeta_\alpha\}$ are the positions of the vortices.

Contrast with conventional case

- In a conventional superfluid the vortices and normal fluid – Bogoliubov quasiparticles - are distinct. (Solenoidal and irrotational parts of the velocity field.)
- In this case since the state is *uniquely* defined by the vortex positions, there are no excitations associated with non-irrotational flow.

No conventional normal fluid

Hamiltonian in vortex representation

- We use $E = \dfrac{\langle\zeta|\mathcal{H}|\zeta\rangle}{\langle\zeta|\zeta\rangle}$ as a variational (fully condensed) trial function. The $\{\zeta_\alpha\}$ become variational parameters.

$$\mathcal{H} = \sum_{i=1}^{N}\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}r_i^2 + \frac{1}{2}\eta\sum_{j=1,\neq i}^{N}\delta(\mathbf{r}_i - \mathbf{r}_j) - \omega.\mathbf{L}\right)$$

- Express energy in terms of the Symmetric Polynomials

$$P_n(\zeta) = \sum_{i_1<i_2<\cdots<i_n} \zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_n}$$

- Consider the incompressible case in a container of radius R (neglecting images)

$$\mathcal{H} = -\frac{1}{2}\Gamma^2\rho\sum_{i<j}\ln|\zeta_i - \zeta_j| - \omega\Gamma\rho\sum_i(R^2 - |\zeta_i|^2)$$

Contrast LLL result

$$E = \pi N S^{-1}\sum_m^M |P_{M-m}(\zeta)|^2(m+1)! - \pi N\omega S^{-1}\sum_m^M |P_{M-m}(\zeta)|^2 m m!$$

$$+\frac{\lambda}{4}N(N-1)S^{-2}\sum_{m,n,p,q=0}^M P^*_{M-m}(\zeta)P^*_{M-n}(\zeta)P_{M-p}(\zeta)P_{M-q}(\zeta)(p+q)!2^{-(p+q)}\delta_{m+n,p+q}$$

$$S = (\pi\sum_{n=0}^{N_v} P^*_{Nv-m}(\zeta)P^*_{Nv-m}(\eta)m!)$$

- Multivortex interaction is analytic in the vortex co-ordinates
- The rotation terms couple to collective variables

Vortex polygons and surface waves

- Consider solutions of the form

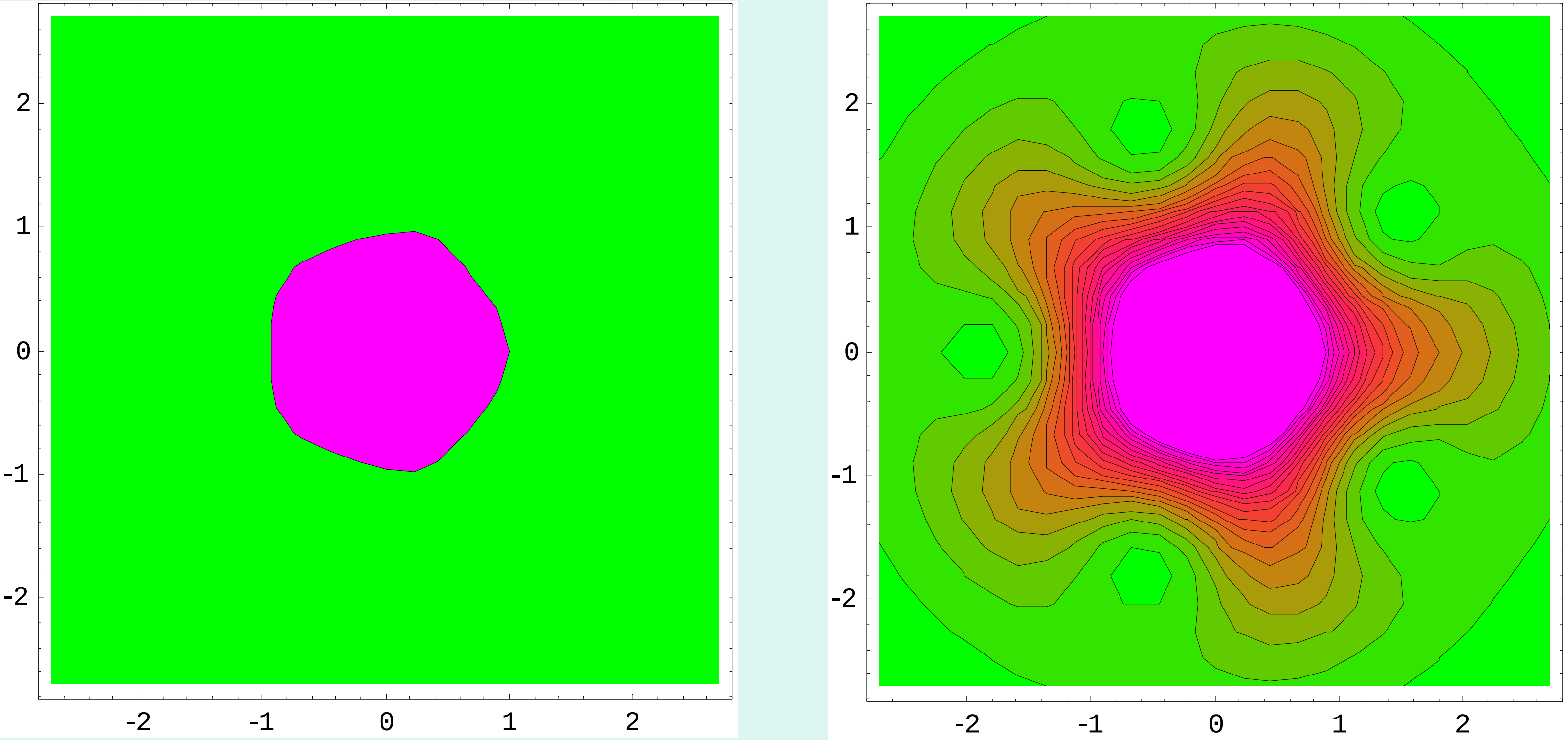
$$\psi(z, z^*, t) = (a_0(t) + a_m(t)z^m)\,\mathrm{e}^{-|\mathrm{z}|^2/2}$$

- There are *m* roots, so *m* vortices in a regular polygon

Relationships between surface waves and vortices in the TF case were realised by

Tsubota, Kasamatsu & Ueda PRA 023603 (2002)
Anglin PRL 87 240401 (2001)

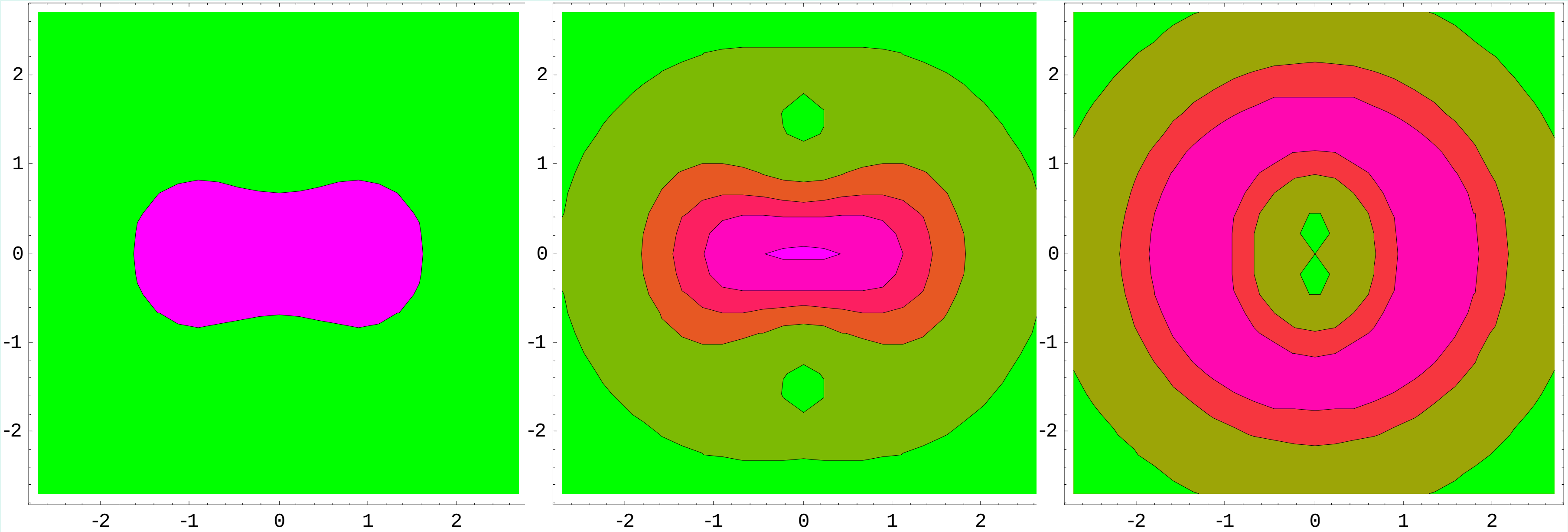
m=5 a_5=0.001



- If one neglects small scale detail – surface wave
- Look more closely see the vortices responsible
- As the vortices move in….

Two vortices

- Now we will examine the dynamics of the vortices as evolve from representing a surface wave to moving under each other's influence at small separation within the trap – for simplicity consider 2 vortices.



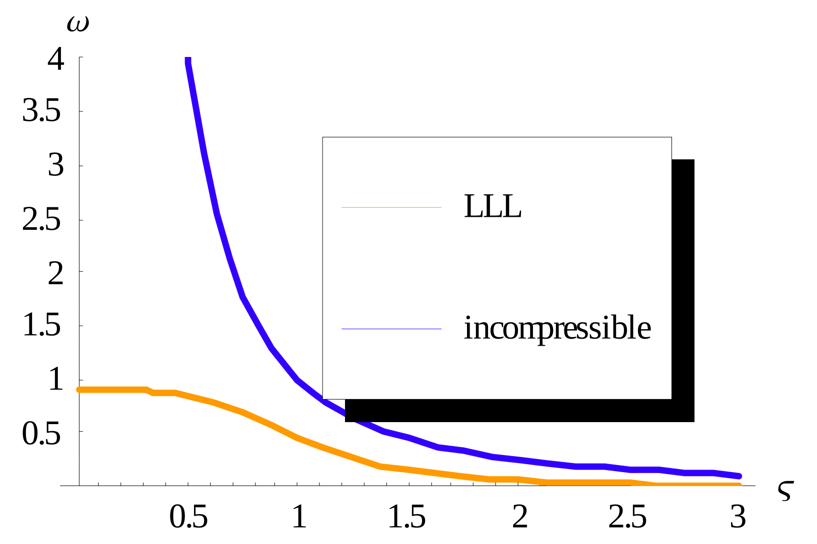
LLL versus incompressible vortex dynamics

Compare LLL frequency for the relative motion of two vortices at separation ζ

$$\omega(2) = 2(1-\omega) - \frac{\lambda N}{4\pi}\left(1 - \frac{3}{4}\frac{2}{(2+\zeta^4)}\right)$$

With the incompressible result:

$\omega = \Gamma/(2\pi\zeta^2)$



Normal Fluid ?

- Although LLL wave-functions completely specified by vortices one could choose to divide them into those inside and outside the trap
- Outside: treated collectively as the surface waves - and treat them as the normal fluid – we have shown that more than one vortex can always feed energy into surface waves.

Elusive physical clarity

- The incompressible limit of superfluid dynamics also eliminates phonons.
- There one can summarise resulting dynamics by the mutual advection of vortices.
- No description of such transparency exists for LLL hydrodynamics.

LLL turbulence

- The properties of small numbers, *n*, of vortices (*n* > 3) in an incompressible superfluid have been studied as chaotic systems. (Eg Aref H, *Ann Rev Fluid Mech* **15** 345 (1983).)
- There is no understanding of the LLL counterparts.
- The nature of the resulting turbulence as $n \to \infty$ has not been explored.

Quantum vortices

Bourne Gunn Wilkin *Phys Rev A* **76**, 053602 (2007)

- Quantisation of one LLL vortex may be performed. But for #>1 .. (cf difficult “classical” mechanics - hydrodynamics.)
- Worse: since angular momentum carried by vortex depends on distance from trap centre, eigenstates anticipated to be superpositions of different #s vortices.
- Nevertheless, can one see vortices in the ψ’s?

Correlations with condensates

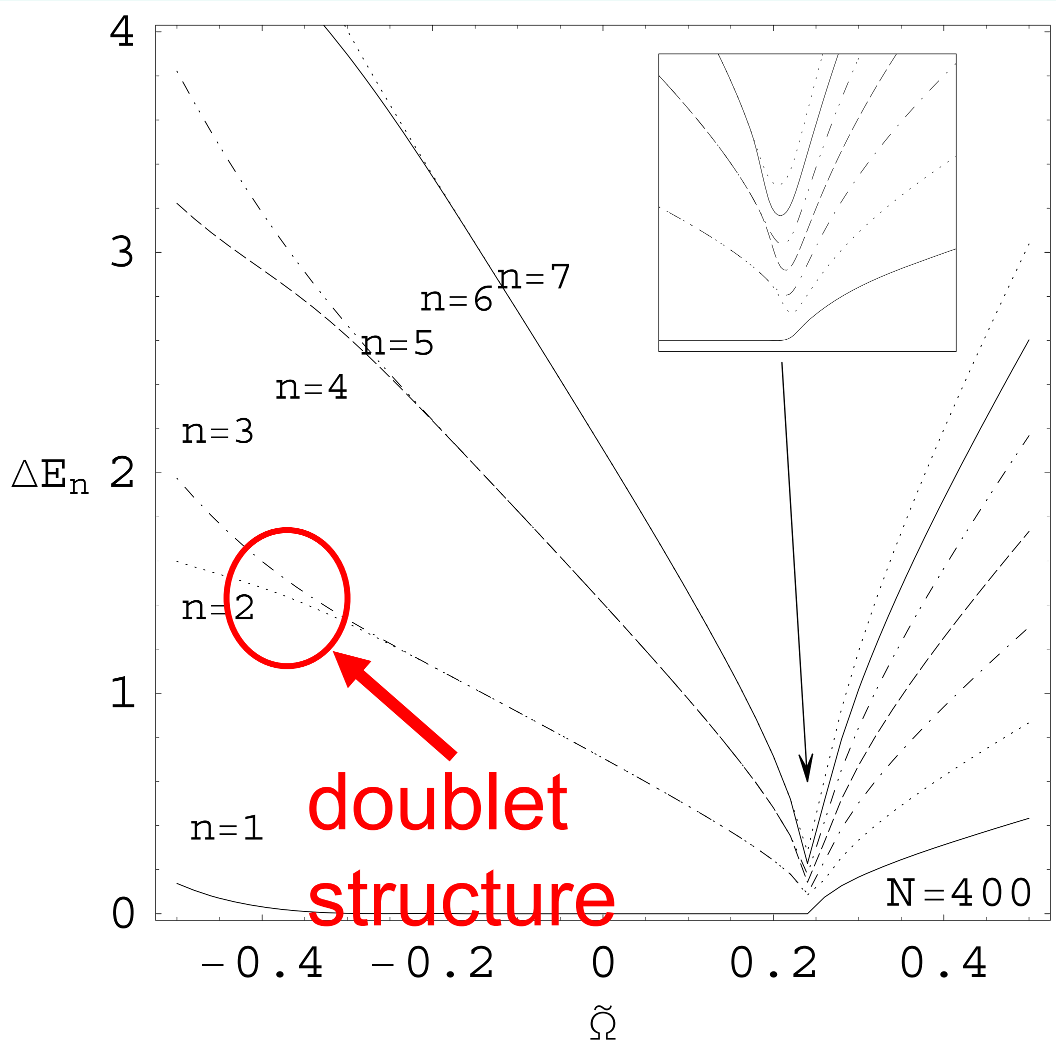
- As well as the correlated ground states, with complete fragmentation, are there cases where the destruction is less complete, but more than just “Bogoliubov”?
- An example is where vortices stop being classical in the LLL.

A surprise

Parke, M.I, Wilkin N.K. et al *Phys. Rev. Lett*. **101** 110401 (2008)

- Entry of first vortex into an elliptical condensate.
- The exact eigenstates may be calculated for weak coupling in LLL – mixing a degenerate manifold of L=0,2,3,….,N.

Energy levels…



doublet structure

ED for *N*=400

Energy gaps measured from GS

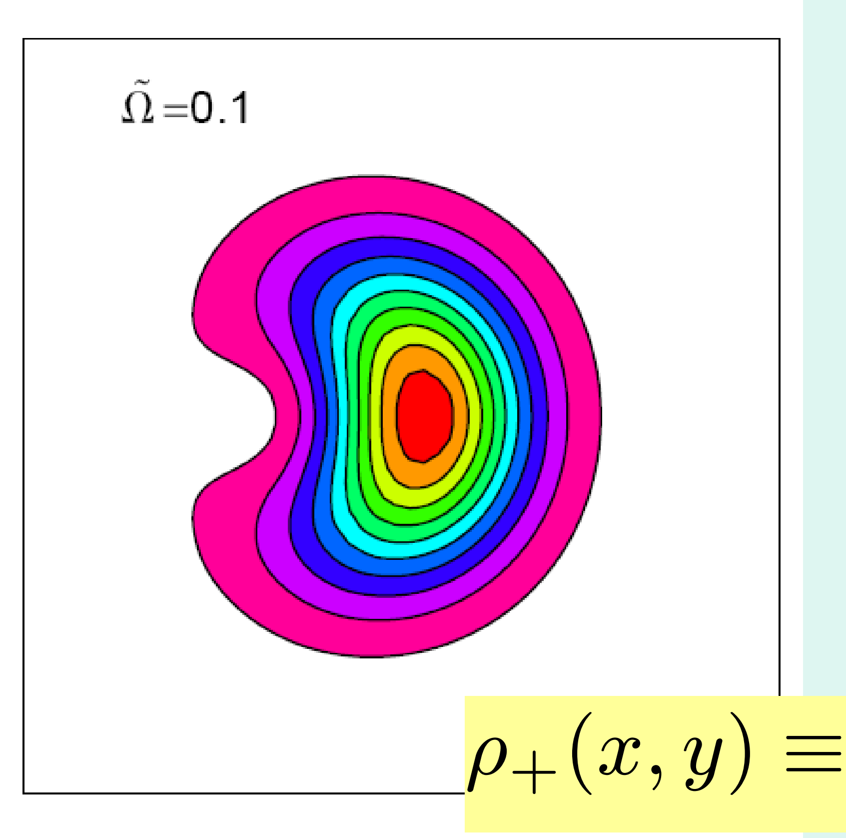
$$\Delta E_1 \to 0$$

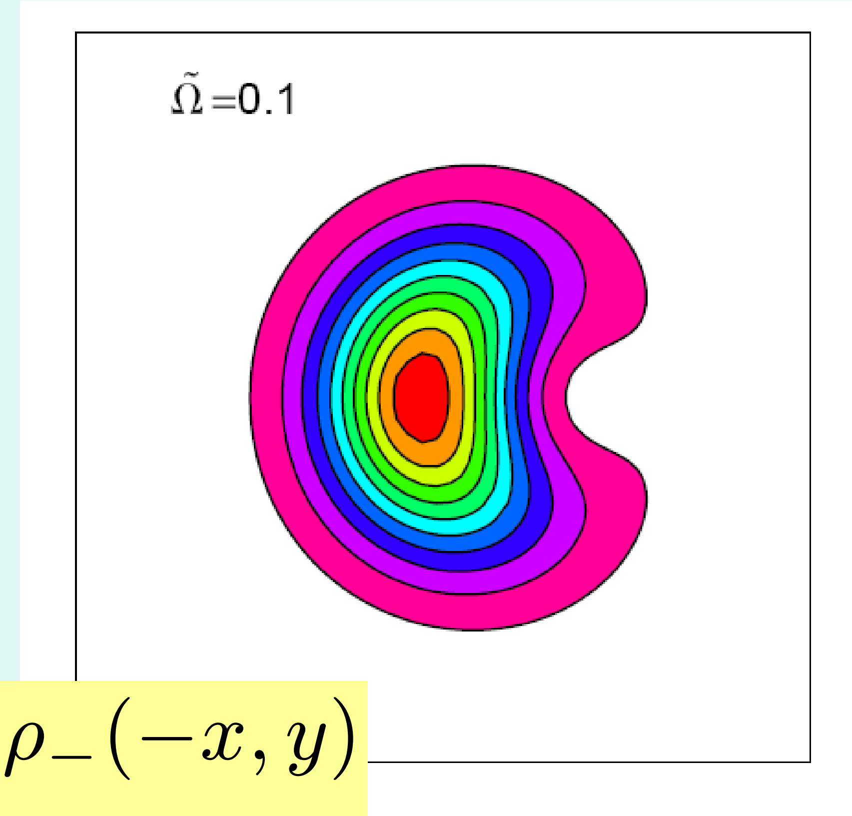
Tunneling?

Pictorial evidence for tunneling

Form the wavefunctions $\psi_\pm = \frac{1}{2}(\psi_0 \pm \psi_1)$

Look at spdm $\rho_\pm$ associated with $\psi_\pm$





$$\rho_+(x,y) \equiv \rho_-(-x,y)$$

Nature of Classical two-vortex Minima

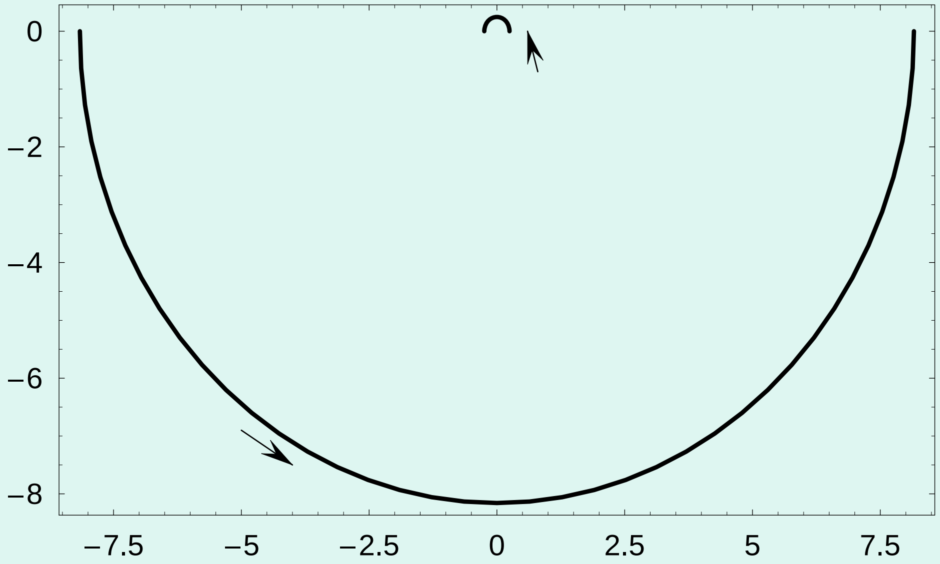
- In terms of vortex positions correspond to
- Two vortices either side of the y-axis.
- Initially (as $\omega \rightarrow \omega_c$) they move inwards symmetrically (quadrupolar deformation)
- Symmetry breaking then occurs, one continues to centre, other heads out to infinity.

Mesoscopics – N^{-1} as $\hbar$

Parke, M.I, Wilkin N.K. et al *Phys. Rev. Lett*. **101** 110401 (2008)

- The classical picture may be quantised, with the particle number entering as the reciprocal "Planck's constant".
- However the tunnelling that results between the classical minima is not single-vortex.

Cooperative Tunneling trajectory



To obtain this return to vortex variables ζ

Vortices have semi circular trajectories, there is a second combination, with the major trajectory above the axis.

Quantum relaxation

- Relaxation of LLL vortex configurations at finite *N* is expected to be quantum mechanical.
- One can hope to see quantum annealing, with a tunable quantum aspect through variation of *N* in future mesoscopic experiments.

Conclusions

- Before coming to correlated phases in rotating gases there are two more accessible experimental (and theoretical) challenges:
- Firstly, to observe and understand the dynamics of LLL “vortex liquids”.
- Secondly to observe quantum annealing with a tunable Planck’s constant given by $1/N$.