Entangled Bose-Einstein Condensation

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KITPC/ITP-CAS, 28/9/2009

BEC as a quantum amplifier

Bose-Einstein condensation amplifies quantum behavior of individual particles to macroscopic phases.

Simplest BEC

$\psi \approx \phi(\mathbf{r}_1)\cdots\phi(\mathbf{r}_N)$ $|\psi\rangle = \frac{1}{\sqrt{N}}(a^\dagger)^N|0\rangle$

All atoms occupy the same single particle states $\phi(\mathbf{r})$.

Mean field theory.

$\phi(\mathbf{r})$ becomes the order parameter [equivalently, $\sqrt{N}\phi(\mathbf{r})$].

Two-component BEC

A mixture of A-atoms and B-atoms:

$$\psi \approx \phi_a(\mathbf{r}_{a1})\cdots\phi_a(\mathbf{r}_{aN_a}) \otimes \phi_b(\mathbf{r}_{b1})\cdots\phi_b(\mathbf{r}_{bN_b})$$

$$|\psi\rangle = \frac{1}{\sqrt{N_a N_b}}(a^\dagger)^{N_a}(b^\dagger)^{N_b}|0\rangle = |\psi\rangle_a \otimes |\psi\rangle_b$$

A-atoms and B-atoms condense separately .

Josephson effect

- $$\psi \approx \phi(\mathbf{r}_1) \cdots \phi(\mathbf{r}_N)$$

$$\phi(\mathbf{r}) = \phi_L(\mathbf{r}) + \phi_R(\mathbf{r})$$

or

$$\phi(\mathbf{r}) = \phi_\uparrow(\mathbf{r})|\uparrow\rangle + \phi_\downarrow(\mathbf{r})|\downarrow\rangle.$$

- The superposition feature of individual particles is amplified to macroscopic behavior.

Spin-1 BEC

- Mean Field: $$\psi \approx \phi(\mathbf{r}_1)\cdots\phi(\mathbf{r}_N)$$

$$\phi(\mathbf{r}) = \phi_1|1\rangle + \phi_0|0\rangle + \phi_{-1}|-1\rangle$$

- Exact (AF):

$$|\psi\rangle \sim [(a_0^\dagger)^2 - 2a_{-1}^\dagger a_1^\dagger]^{N/2}|0\rangle$$

$$\psi \approx \mathcal{S}\{\phi(\mathbf{r}_1,\mathbf{r}_2)\cdots\phi(\mathbf{r}_{N-1},\mathbf{r}_N)\}$$

$$\phi(\mathbf{r}_1,\mathbf{r}_2) \sim \phi_{00}|0,0\rangle - \sqrt{2}\phi_{-1,1}(|-1,1\rangle + |1,-1\rangle).$$

Condensation in a two-particle superposition state of identical particles.

Can BEC amplify entanglement between individual distinguishable particles to a macroscopic phase?

Quantum entanglement

$$|\Psi\rangle \neq |\psi\rangle_A \otimes |\psi\rangle_B$$

- Most (or only?) essential difference between quantum mechanics and classical mechanics.
- A and B parts are distinguishable.
- For our purpose, A and B are systems of identical particles.

2 distinguishable species × 2 internal states (pseudospins)

- Each atom can flip the spin between two states, but cannot transit between the atom species.
- $N_{i\uparrow}$ and $N_{i\downarrow}$ $(i = a, b)$ are not conserved. $N_i = N_{i\uparrow} + N_{i\downarrow}$ is strictly conserved.

BEC occuring in an interspecies two-particle entangled state

YS, Int. J. Mod. Phys. B 15, 3007 (2001)

$$\Psi = \mathcal{N} S[\phi(\mathbf{r}_{a1}, \mathbf{r}_{b1}) \cdots \phi(\mathbf{r}_{aN}, \mathbf{r}_{bn})].$$

$$\phi(\mathbf{r}_a, \mathbf{r}_b) = \frac{1}{\sqrt{2}}[\phi_{a\uparrow}(\mathbf{r}_a)|\uparrow\rangle_a \phi_{b\downarrow}(\mathbf{r}_b)|\downarrow\rangle_b - \phi_{a\downarrow}(\mathbf{r}_a)|\downarrow\rangle_a \phi_{b\uparrow}(\mathbf{r}_b)|\uparrow\rangle_b]$$

- $\phi(\mathbf{r}_a, \mathbf{r}_b)$ is the entangled order parameter.
- Let's call it entangled BEC (EBEC). It is the BEC amplification of entanglement.
- Could be a useful source of EPR pairs.
- Second quantization form:

$$|G_0\rangle = (\sqrt{N+1}N!)^{-1}(a_\uparrow^\dagger b_\downarrow^\dagger - a_\downarrow^\dagger b_\uparrow^\dagger)^N|0\rangle$$

SU(2) symmetric model

Kuklov&Svistunov, PRL 89, 170403 (2002)

$$H = \int d\mathbf{x}[\psi_\sigma H_1 \psi_\sigma + \frac{g}{2}\psi_\sigma^\dagger \psi_{\sigma_1}^\dagger \psi_{\sigma_1} \psi_\sigma].$$

- To conserve an initial small S during cooling, the atoms occupy two orbital modes.
- Atom numbers in the two orbitals are not conserved, but are fixed through measurement or microcanonical distribution.
- For such fixed numbers, the "ground state" is apparently similar to EBEC, with two species replaced as the two orbital modes.
- But all the atoms are identical particles. If the atoms are taken out of the trap, the orbital identity is lost. So it is more like spin-1 BEC: BEC of two-particle superposition state, with three single-particle degrees of freedom (spin, orbital mode, position).

Our second quantized Hamiltonian

YS& Q. Niu, PRL. 96, 140401 (2006)

Only the lowest orbital modes are occupied.
The two species are distinguishable, can have a small total spin.

$$\mathcal{H} = \mathcal{H}_a + \mathcal{H}_b + \sum_{\sigma\sigma'} K^{(ab)}_{\sigma\sigma'} N_{a\sigma} N_{b\sigma'} + \frac{K_e}{2}(a^\dagger_\uparrow a_\downarrow b^\dagger_\downarrow b_\uparrow + a^\dagger_\downarrow a_\uparrow b^\dagger_\uparrow b_\downarrow)$$

$$\mathcal{H}_j = f_{j\sigma} N_{j\sigma} + \tfrac{1}{2}\sum_{\sigma\sigma'} K^{(jj)}_{\sigma\sigma'} N_{j\sigma} N_{j\sigma'},$$

Ks are proportional to corresponding scattering lengths.

K_e (spin exchange) term causes interspecies entanglement in the ground state

Spin representation

$$\mathbf{S}_a = \sum_{\sigma,\sigma'} a^\dagger_\sigma \mathbf{s}_{\sigma\sigma'} a_{\sigma'}, \; \mathbf{S}_b = \sum_{\sigma,\sigma'} b^\dagger_\sigma \mathbf{s}_{\sigma\sigma'} b_{\sigma'}$$

- The Hamiltonian becomes that of two giant spins $S_a = N_a/2$ and $S_b = N_b/2$

$$\frac{\mathcal{H}}{J_z} = \frac{K_e}{J_z}(S_{ax}S_{bx} + S_{ay}S_{by}) + S_{az}S_{bz} + B_a S_{az} + B_b S_{bz} + C_a S_{az}^2 + C_b S_{bz}^2 + \frac{E_0}{J_z}$$

- Coefficients are functions of K’s.

Isotropic parameter point

$$\mathcal{H} = J_z \mathbf{S}_a \cdot \mathbf{S}_b$$

- Ground states:

$$|G_{S_z}\rangle = A(a_\uparrow^\dagger)^{n_\uparrow}(a_\downarrow^\dagger)^{n_\downarrow}(a_\uparrow^\dagger b_\downarrow^\dagger - a_\downarrow^\dagger b_\uparrow^\dagger)^{N_b}|0\rangle$$

$$n_\uparrow = N_a/2 - N_b/2 + S_z,\ n_\downarrow = N_a/2 - N_b/2 - S_z$$

Degenerate for $S_z \neq 0$, unique for each S_z .

Non-degenerate for $S_z = 0$

For $N_a = N_b = N$:

$$|G_0\rangle = (\sqrt{N+1}N!)^{-1}(a_\uparrow^\dagger b_\downarrow^\dagger - a_\downarrow^\dagger b_\uparrow^\dagger)^N|0\rangle$$

EBEC is the exact ground state

Entanglement lowers the energy

A simple example:

$$h(\mathbf{r}_a) + h(\mathbf{r}_b) + U_1(\mathbf{r}_a - \mathbf{r}_b) + U_2(\mathbf{r}_a - \mathbf{r}_b)(|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|)$$

$$U_2 > 0$$

$$\phi_a(\mathbf{r}_a)\phi_b(\mathbf{r}_b)(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)/\sqrt{2}$$

has lower energy than

$$\phi_a(\mathbf{r}_a)\phi_b(\mathbf{r}_b)|\sigma\rangle|\sigma'\rangle$$

EBEC is a special kind of fragmented BEC

$$\rho_a(\mathbf{r},\mathbf{r}') = (N_a/2+S_z)\phi^*_{a\uparrow}(\mathbf{r}')\phi_{a\uparrow}(\mathbf{r}) + (N_a/2-S_z)\phi^*_{a\downarrow}(\mathbf{r}')\phi_{a\downarrow}(\mathbf{r}),$$
$$\rho_b(\mathbf{r},\mathbf{r}') = N_b/2\phi^*_{b\uparrow}(\mathbf{r}')\phi_{b\uparrow}(\mathbf{r}) + N_b/2\phi^*_{b\downarrow}(\mathbf{r}')\phi_{b\downarrow}(\mathbf{r}).$$

All a atoms form a fragmented condensate,

while all b atoms form another one.

- In an entangled particle pair, each particle is not in a pure quantum state, but in a mixed state. Likely, in EBEC, each species of atoms does not undergo a simple BEC, but form a fragmented condensate.
- There is phase coherence among interspecies pairs. The pairs undergo simple BEC.
- Interspecies entanglement can be regarded as a characterization of the non-mean field nature of single particles, or deviation from the usual multicomponent BEC.
- Closely associated with symmetry.

Quantifying the entanglement

- Quantified as entanglement entropy: von Neumann entropy of the reduced density matrix of each species.
- The calculation is most easily done by using occupation number representations

  [method: YS, PRA 67, 024301 (03)].
- The entanglement is maximal (1) at |G0>.

How this entanglement and fragmentation survive the coupling anisotropy and the nonvanishing of B_a, B_b, C_a, C_b

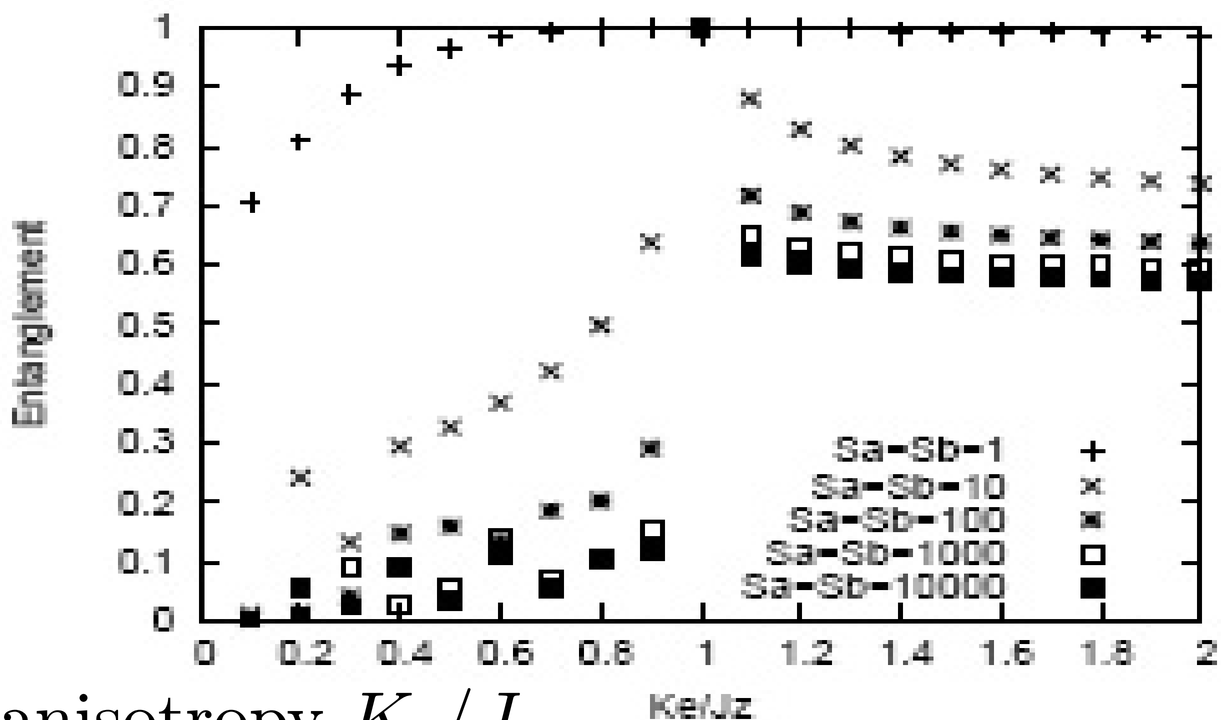
$$\frac{\mathcal{H}}{J_z} = \frac{K_e}{J_z}(S_{ax}S_{bx} + S_{ay}S_{by}) + S_{az}S_{bz} + B_aS_{az} + B_bS_{bz} + C_aS_{az}^2 + C_bS_{bz}^2 + \frac{E_0}{J_z}$$

Numerical investigation

- Use the Lanczos method to find the ground state numerically.
- It is found that in a wide parameter regime, there is significant amount of interspecies entanglement in the ground state.

Persistence of entanglement in a wide parameter regime (1)

- $B_a = B_b = C_a = C_b = 0$, $S_z = 0$

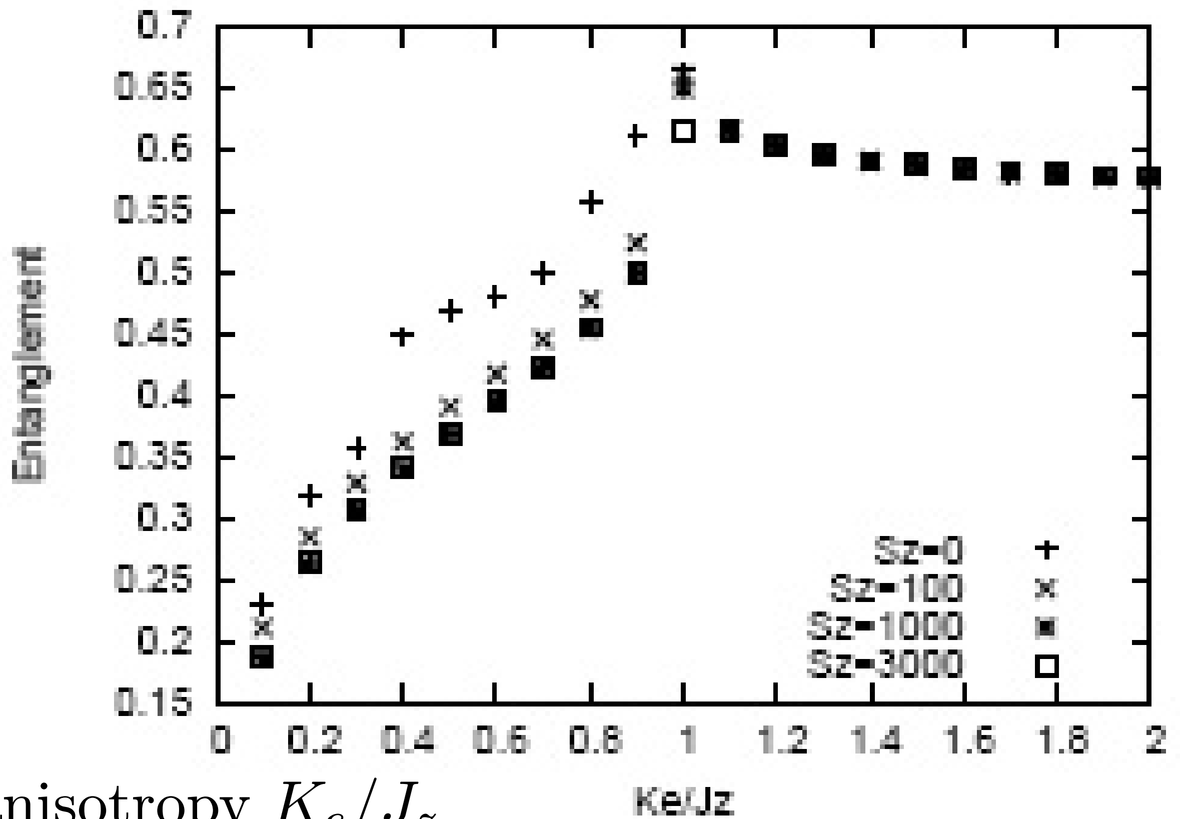


Coupling anisotropy K_e/J_z

Persistence of entanglement in a wide parameter regime (2)

$B_a = B_b = C_a = C_b = 0$

- $S_a = 12000,\ S_b = 10000$



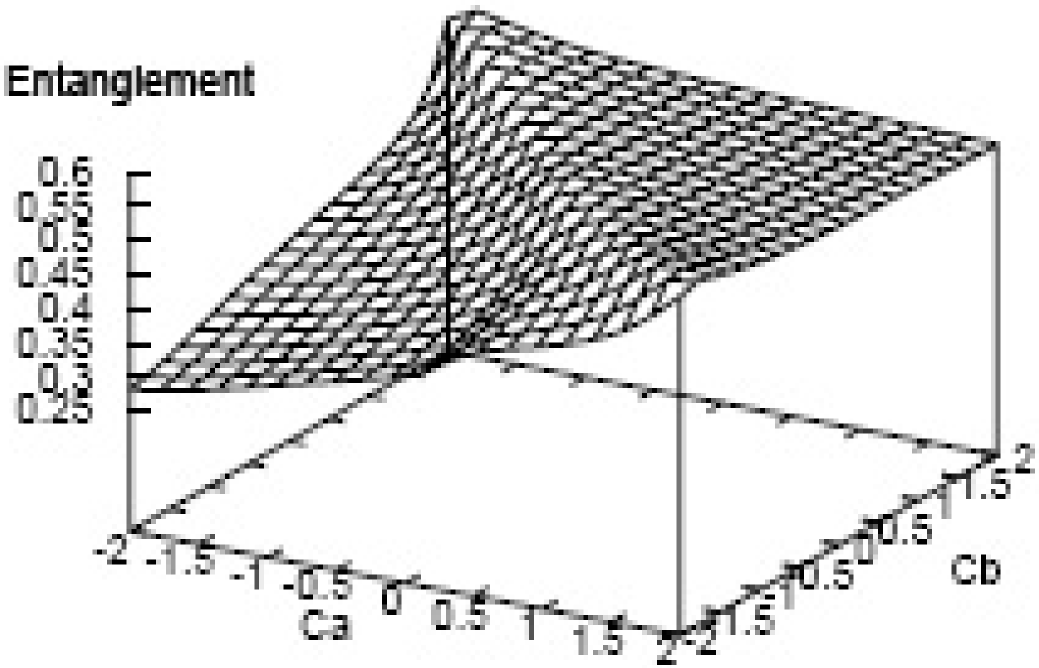
Coupling anisotropy K_e/J_z

Persistence of entanglement in a wide parameter regime (3)

C_a and C_b nonzero, $B_a = B_b = 0$, under typical values

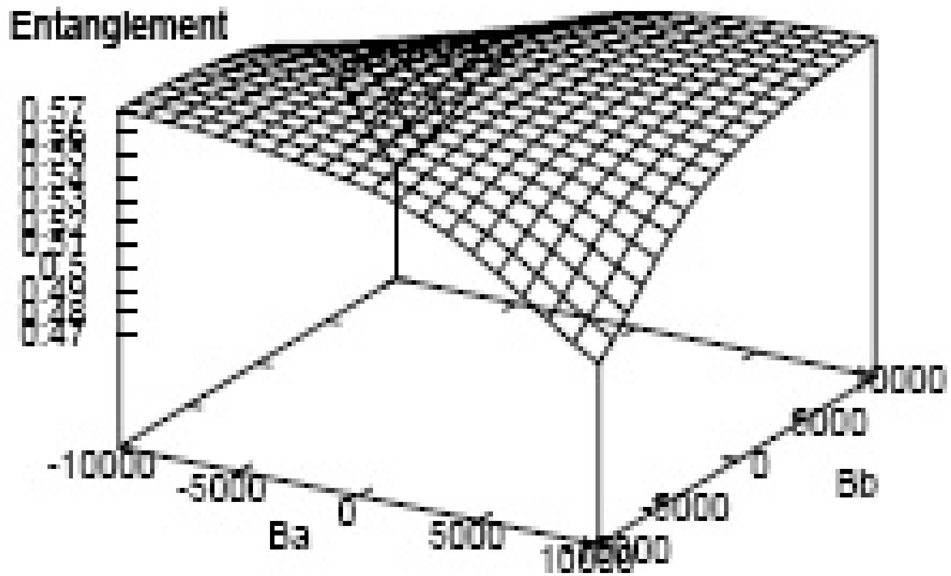
$S_a = 12000$, $S_b = 10000$, $S_z = 1000$, $K_e/J_z = 1.2$

J_z, $C_a J_z$ and $C_b J_z$ are of the same order of magnitude



Persistence of entanglement in a wide parameter regime (4)

Typically choose $C_a = 0.2$ and $C_b = 0.4$



Analytical investigation

- Can symmetry, fragmentation and entanglement persist even under symmetry breaking perturbation?

YS, EPL 86, 60008 (2009)

Perturbed ground state

S is not conserved, but S_z is still conserved.

Focus on $S_z = 0$:

$\mathcal{H}_1$ causes mixing of $|S, S_z = 0\rangle$ with different values of S.

$$|\Psi_n\rangle = \sum_S \psi_n(S)|S,0\rangle$$

$$\psi_0 = (8K_e/\pi\Delta)^{1/4} e^{-\frac{K_e}{\Delta}S^2}.$$

$$\Delta = \sqrt{2K_e f},$$
$$f = (B_b - B_a)N/2 - (C_a + C_b + K_e - J_z)N^2/2.$$

Thus for a finite volume system (K_e remains nonzero), $\psi_0(S) \to \delta(S)$ as $\Delta \to 0$.

A finite-volume converse situation of spontaneous symmetry breaking in an infinite system

- Key factor: the volume is finite.

If take $\Omega \to \infty$ first,
then $\psi_0(S)$ becomes independent of l.
Consequently the ground state is an equal superposition
of $|S, 0\rangle$ of all possible values of S, hence breaks the symmetry.
This is SSB if the symmetry breaking perturbation is infinitesimal.

So the GS approaches the unperturbed entangled BEC state $|0,0\rangle$.
The symmetry is protected by the vanishing gap.

Therefore EBEC can persist in a significant parameter regime.

Detection of the entanglement (1)

- Fluctuations of $N_{i\sigma}$: $\sqrt{\langle N_{a\sigma}^2 \rangle - \langle N_{a\sigma} \rangle^2} / \langle N_{a\sigma} \rangle$
- Can be obtained from density fluctuation, which is self-averaging, and can be studied in a single optical image

$$\rho_{i\sigma}(\mathbf{r}_i) = N_{i\sigma} |\phi_{i\sigma}(\mathbf{r}_i)|^2$$

$$\sqrt{\langle \rho_{i\sigma}(\mathbf{r}_i)^2 \rangle - \langle \rho_{i\sigma}(\mathbf{r}_i) \rangle^2} / \langle \rho_{i\sigma}(\mathbf{r}_i) \rangle = \sqrt{\langle N_{i\sigma}^2 \rangle - \langle N_{i\sigma} \rangle^2} / \langle N_{i\sigma} \rangle$$

Detection of the entanglement (2)

- Nonvanishing of the correlations

$$C_{\sigma,\sigma'} = \langle N_{a\sigma} N_{b\sigma'} \rangle - \langle N_{a\sigma} \rangle \langle N_{b\sigma'} \rangle$$

$$g(\mathbf{r}_a, \sigma; \mathbf{r}_b, \sigma') = \langle \rho_{a\sigma}(\mathbf{r}_a) \rho_{b\sigma'}(\mathbf{r}_b) \rangle - \langle \rho_{a\sigma}(\mathbf{r}_a) \rangle \langle \rho_{b\sigma'}(\mathbf{r}_b) \rangle$$

$$g(\mathbf{r}_a, \sigma; \mathbf{r}_b, \sigma') / \langle \rho_{a\sigma}(\mathbf{r}_a) \rangle \langle \rho_{b\sigma'}(\mathbf{r}_b) \rangle = C_{\sigma,\sigma'} / \langle N_{a\sigma} \rangle \langle N_{b\sigma'} \rangle$$

Detection of entanglement (3)

- Measuring spin of an A-atom,

$$P_\sigma = \langle a^\dagger_\sigma a_\sigma \rangle / \sum_{\sigma'} \langle a^\dagger_{\sigma'} a_{\sigma'} \rangle$$

- Joint measurement of the spins of an A-atom and a B-atom which leave the trap

$$P_{\sigma,\sigma'} = \langle b^\dagger_{\sigma'} a^\dagger_\sigma a_\sigma b_{\sigma'} \rangle / \sum_{\sigma_a,\sigma_b} \langle b^\dagger_{\sigma_b} a^\dagger_{\sigma_a} a_{\sigma_a} b_{\sigma_b} \rangle.$$

$$P_{\sigma_a,\sigma_b} \neq P_{\sigma_a} P_{\sigma_b}$$

Experimental feasibility

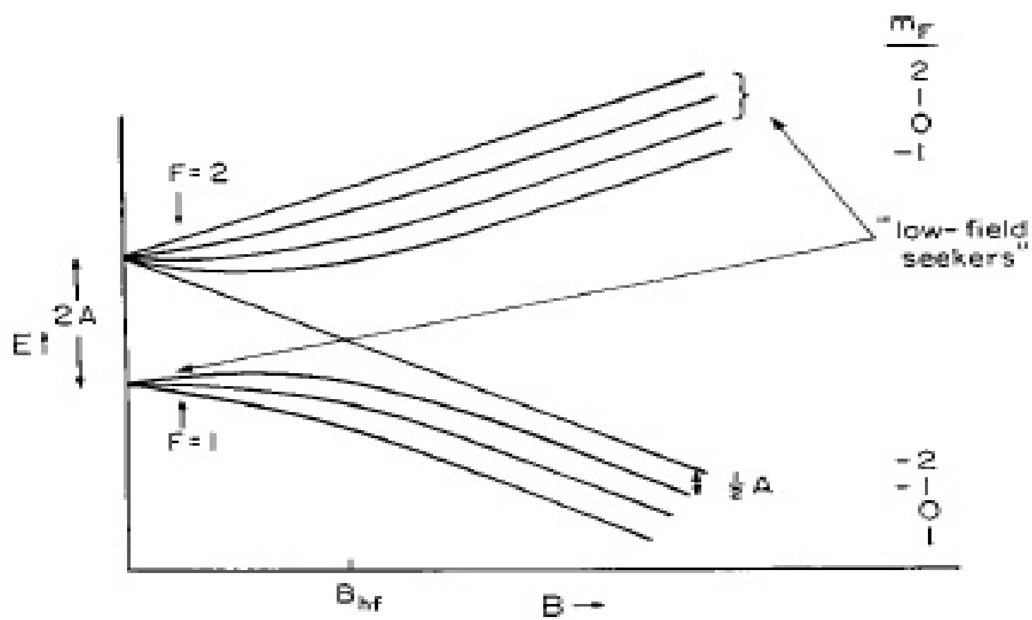
YS, EPL 86, 60008 (2009)

Use two species of alkali atoms in an optical trap.
Use $|F=2, m_F=2\rangle$ and $|F=1, m_F=1\rangle$
to represent the two pseudospin states.

Energy splitting and total F and m_F conservation forbid scattering to other hyperfine states.

Recent two-componet Experiments [Thalhammer et al

PRL 08; Papp et al, PRL 08]
can be extended for our purpose.



Dynamics

YS, arXiv:0909.4990

Time-dependent Gross-Pitaevskii equations

$\delta \int_{t_1}^{t_2} L dt = 0 \longrightarrow$

$$
\begin{aligned}
i\hbar\frac{\partial \phi_{j\sigma}(\mathbf{r})}{\partial t} = \{ & -\frac{\hbar^2}{2m_j}\nabla^2 + U_j(\mathbf{r}) + \frac{2(N-1)}{3} g_{\sigma\sigma}^{(jj)} |\phi_{j\sigma}(\mathbf{r})|^2 + \frac{N-1}{3} g_{\sigma\bar{\sigma}}^{(jj)} |\phi_{j\bar{\sigma}}(\mathbf{r})|^2 \\
& + \frac{N-1}{3} g_{\sigma\sigma}^{(j\bar{j})} |\phi_{\bar{j}\sigma}(\mathbf{r})|^2 + \frac{2N+1}{3} g_{\sigma\bar{\sigma}}^{(j\bar{j})} |\phi_{\bar{j}\bar{\sigma}}(\mathbf{r})|^2 \} \phi_{j\sigma}(\mathbf{r}) \\
& - \frac{N+2}{6} g_e \phi_{\bar{j}\bar{\sigma}}^*(\mathbf{r}) \phi_{\bar{j}\sigma}(\mathbf{r}) \phi_{j\bar{\sigma}}(\mathbf{r}),
\end{aligned}
$$

Supercurrents

$$n_{j\sigma} = (N/2)\phi^*_{j\sigma}\phi_{j\sigma},$$

$$\mathbf{J}_{j\sigma} = \frac{\hbar}{2mi}\frac{N}{2}(\phi^*_{j\sigma}\nabla\phi_{j\sigma} - \nabla\phi^*_{j\sigma}\phi_{j\sigma}) = n_{j\sigma}\mathbf{v}_{j\sigma}$$

$$\frac{\partial n_{j\sigma}(\mathbf{r},t)}{\partial t} + \nabla\cdot\mathbf{J}_{j\sigma}(\mathbf{r},t) = S_{j\sigma}$$

$$S_{a\sigma} = -S_{b\sigma} = -S_{a\bar{\sigma}} = S_{b\bar{\sigma}}$$

$$\frac{\partial(n_{j\uparrow}+n_{j\downarrow})}{\partial t} + \nabla\cdot(\mathbf{J}_{j\uparrow} + \mathbf{J}_{j\downarrow}) = 0.$$

$$\frac{\partial(n_{a\sigma}+n_{b\sigma})}{\partial t} + \nabla\cdot(\mathbf{J}_{a\sigma} + \mathbf{J}_{b\sigma}) = 0.$$

$$\frac{\partial}{\partial t}\sum_{j,\sigma} n_{j\sigma} + \nabla\cdot\sum_{j,\sigma}\mathbf{J}_{j\sigma} = 0.$$

Spin supercurrent

$$n_j^s = m_F(j,\uparrow)n_{j\uparrow} + m_F(j,\downarrow)n_{j\downarrow},$$

$$\mathbf{J}_j^s = m_F(j,\uparrow)\mathbf{J}_{j\uparrow} + m_F(j,\downarrow)\mathbf{J}_{j\downarrow},$$

$$\frac{\partial n_j^s}{\partial t} + \nabla \cdot J_j^s = [(m_F(j,\uparrow) - m_F(j,\downarrow)]S_{j\uparrow}.$$

$$\frac{\partial (n_a^s + n_b^s)}{\partial t} + \nabla \cdot (J_a^s + J_b^s) = 0,$$

The total spin supercurrent is conserved.

Hydrodynamics

$$\phi_{j\sigma} = f_{j\sigma} e^{i\Phi_{j\sigma}}, \qquad \mathbf{v}_{j\sigma} = \frac{\hbar}{m}\nabla\Phi_{j\sigma},$$

$$m\frac{\partial \mathbf{v}_{j\sigma}}{\partial t} = -\nabla(\tilde{\mu}_{j\sigma} + \tfrac{1}{2}m_j v_{j\sigma}^2),$$

$$\tilde{\mu}_{j\sigma} = -\frac{\hbar^2}{2m_j\sqrt{n_{j\sigma}}}\nabla^2\sqrt{n_{j\sigma}} + U_{j\sigma} + \frac{2(N-1)}{3}g_{\sigma\sigma}^{(jj)}n_{j\sigma} + \frac{(N-1)}{3}g_{\sigma\bar{\sigma}}^{(jj)}n_{j\bar{\sigma}} + \frac{(N-1)}{3}g_{\sigma\sigma}^{(j\bar{j})}n_{\bar{j}\sigma} + \frac{(2N+1)}{3}g_{\sigma\bar{\sigma}}^{(j\bar{j})}n_{\bar{j}\bar{\sigma}} + V_{j\sigma}^{(e)},$$

$$V_{j\sigma}^{(e)} = \frac{N+2}{6}g_e\sqrt{\frac{n_{\bar{j}\bar{\sigma}}n_{\bar{j}\sigma}n_{j\bar{\sigma}}}{n_{j\sigma}}}\cos(\Phi_{\bar{j}\sigma} + \Phi_{j\bar{\sigma}} - \Phi_{\bar{j}\bar{\sigma}} - \Phi_{j\sigma})$$

is due to spin exchange scattering.

There is inter-dependence between superfluid velocities of different components.

Elementary excitations

Assume: $U_{j\sigma} = 0,\ g^{(jj)}_{\sigma\sigma} = g_j,\ g^{(jj)}_{\sigma\bar{\sigma}} = 2g_j,\ g^{(ab)}_{\sigma\sigma} = g_s,\ g^{(ab)}_{\sigma\bar{\sigma}} = g_d.$

Variation from the wavefunctions at ground state:

$$\delta\phi_{j\sigma} = \frac{e^{-i\frac{\mu_{j\sigma}t}{\hbar}}}{\sqrt{\Omega}}\left[u_{j\sigma}(q)e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} - v^{*}_{j\sigma}(q)e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\right],$$

$$A\mathbf{U} = \omega\mathbf{U},$$

$$\mathbf{U} = (u_{a\uparrow}, v_{a\uparrow}, u_{a\downarrow}, v_{a\downarrow}, u_{b\uparrow}, v_{b\uparrow}, u_{b\downarrow}, v_{b\downarrow})^{T}.$$

Elementary excitations (continued)

Four excitations (i=1,2,3):

Three are Bogoliubov-like, two of which depend on both intra- and interspecies scattering, the other depends only on interspecies scattering,

$$E_q^{(i)} = [E_q^{(0)}(E_q^{(0)} + \Gamma^{(i)})]^{1/2}, \qquad E_q^0 = \hbar^2 q^2 / 2m$$

The fourth excitation is a gapped mode,

$$E_q^{(4)} > 0 \text{ when } q \to 0.$$

Summary

- Entangled BEC: Bose-Einstein condensation, occurring in an interspecies two-particle entangled state.
- The (macroscopic) order parameter is entangled.
- Like a pure two-particle entangled state, where each particle is not in any pure spin state, there is no simple BEC of either species; there is only a global simple BEC.
- The entangled BEC can persists in wide parameter regime, as far as the energy gap remains vanishing, and can be experimentally realized.
- The system may be useful as a source of entangled atom pairs.
- There are four supercurrents and four spin supercurrents. The total ones are conserved.
- There are three Bogoliubov-like modes and a gapped mode.

Thank you for your attention!