The normal state of Fermi gases

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Outline

1. Introduction
2. Ideal Fermi gas (Homogeneous, trapped)
3. Dilute Fermi gas
4. Fermi liquid theory

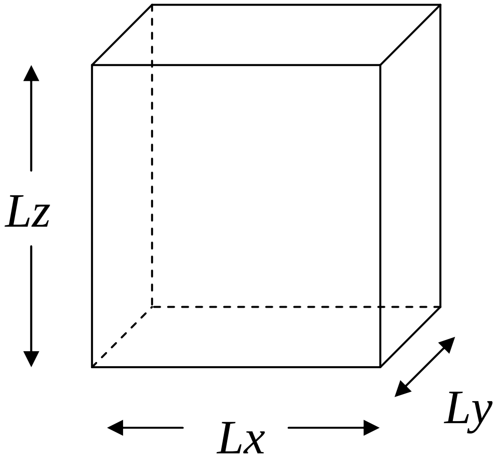
1. Introduction

Topics about Fermi gases:

- Fermi gas in optical lattices (H. Zhai)
- Low-dimensional Fermi gas (S. Chen)
- BEC-BCS crossover (Q.-J. Chen)
- Dipolar Fermi gas (H. Pu)
- Quantum-Hall states in rotational Fermi gas
- Large-spin (S>1/2) Fermi gas
- …

2. Ideal Fermi gas (S=1/2)

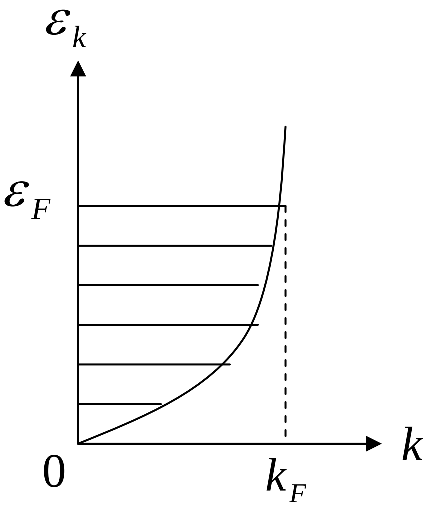
Homogeneous case



Single-particle eigenstate

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{\exp[-i\mathbf{k}\cdot\mathbf{r}]}{\sqrt{L_x L_y L_z}}, \quad \mathbf{k} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \frac{2\pi n_z}{L_z}\right),$$

Eigenenergy $\varepsilon_k = \dfrac{\hbar^2 k^2}{2m}$



Ground state---filled Fermi sphere.

Fermi wavevector $k_F = (3\pi^2 n)^{1/3}$

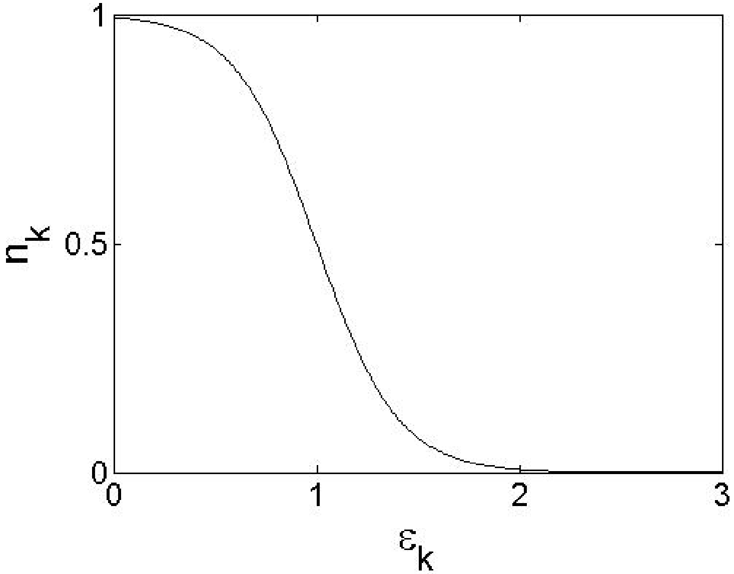
Fermi energy $\varepsilon_F = \dfrac{\hbar^2}{2m} k_F^2$ Fermi Temperature $T_F = \dfrac{\varepsilon_F}{k_B}$

Ground-state Energy density $\varepsilon = \dfrac{\hbar^2 k_F^5}{10\pi^2 m} = \dfrac{\hbar^2 (3\pi^2 n)^{5/3}}{10\pi^2 m}$

Fermi distribution function $$n_k = \frac{1}{\exp[\beta(\varepsilon_k - \mu)] + 1}$$

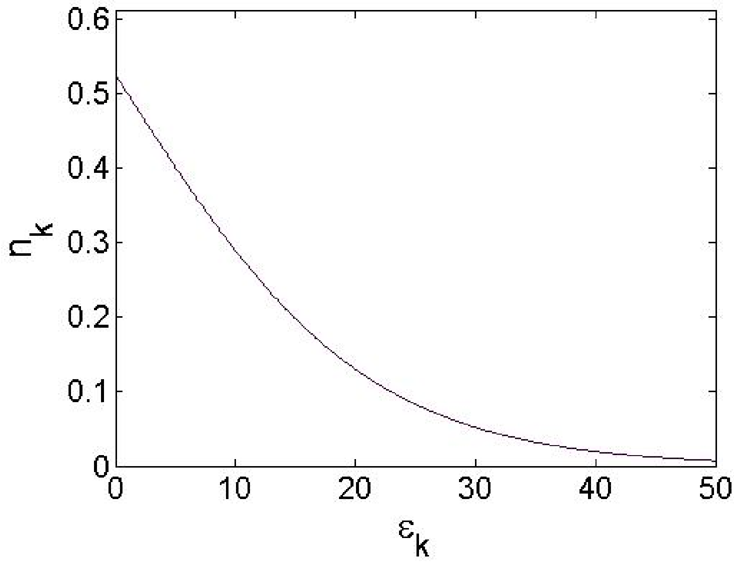
Quantum degeneracy

$$T \le T_F$$



High temperature (classical)

$$T \gg T_F$$



Harmonic trap $V(\mathbf{r}) = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$

Single-particle eigenenergy

$$\varepsilon_n = \hbar\omega_x(n_x + \frac{1}{2}) + \hbar\omega_y(n_y + \frac{1}{2}) + \hbar\omega_z(n_z + \frac{1}{2})$$

(Kinetic energy=Trapping energy)

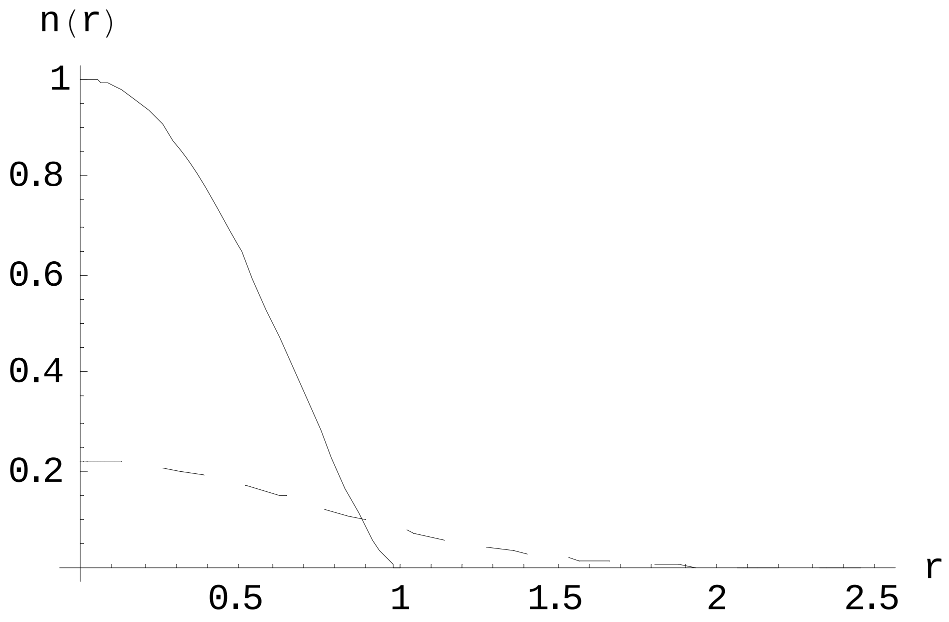
Fermi energy $\varepsilon_F = \hbar(N\omega_x\omega_y\omega_z / 2)^{1/3}$

Thomas-Fermi approximation (T=0)

$$\frac{\hbar^2[3\pi^2 n(\mathbf{r})]^{2/3}}{2m} = \varepsilon_F - V(\mathbf{r})$$

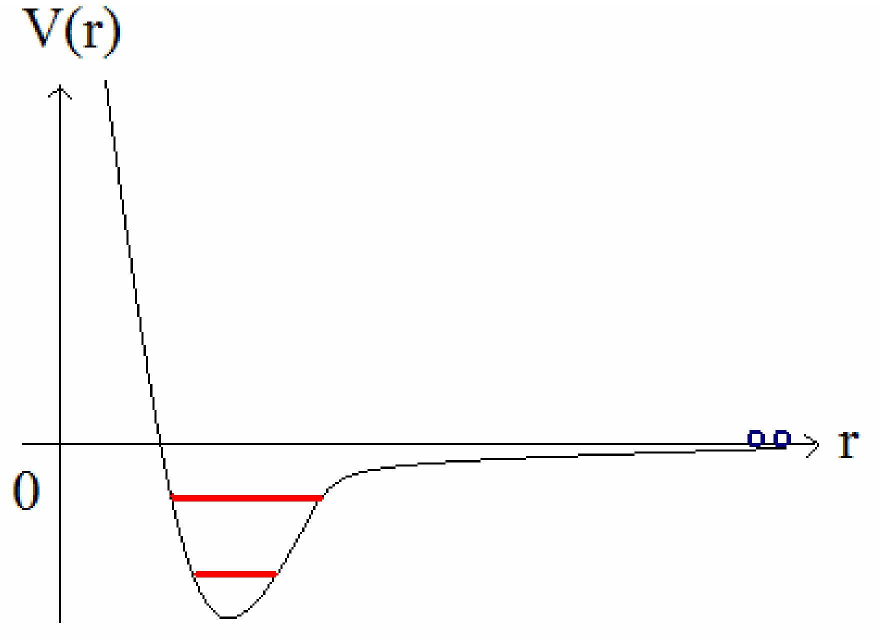
Classical distribution (T>>T$_F$)

$$n(\mathbf{r}) \propto \exp[-\beta V(\mathbf{r})]$$



3. Dilute Fermi gas

Atom-atom interaction



Atom energy $\varepsilon_{\mathbf{k}} = \dfrac{\hbar^2 k^2}{2m}$

Molecule binding energy

$$\lim_{a_s \to +\infty} \varepsilon_0 \approx \frac{\hbar^2}{m a_s^2}$$

S-wave phase shift $\delta_0 \to -k a_s$

Effective potential $V(\mathbf{r}) = 4\pi \dfrac{\hbar^2}{m} a_s \delta^3(r)$

The **effective potential** is proportional to the **scattering length** $\boldsymbol{a_s}$.

Perturbation theory

Hamiltonian $H = H_0 + \delta H, \quad H_0 = -\int d^3 r \sum_\sigma \psi_\sigma^+(\mathbf{r}) \frac{\hbar^2 \nabla^2}{2m} \psi_\sigma(\mathbf{r}),$

$$\delta H = g \int d^3 r \psi_\uparrow^+(\mathbf{r}) \psi_\downarrow^+(\mathbf{r}) \psi_\downarrow(\mathbf{r}) \psi_\uparrow(\mathbf{r}), \quad g = 4\pi \frac{\hbar^2}{m} a_s$$

Unperturbed ground state $\left|\phi_0^{(0)}\right\rangle = \prod_{|\mathbf{k}|<k_F, \sigma} \psi_{\mathbf{k}\sigma}^+ \left|0\right\rangle, \quad E_0^{(0)} = \frac{3}{5} N \varepsilon_F$

First-order perturbation $\left|\phi_0^{(1)}\right\rangle = \sum_{m \neq 0} \frac{\left\langle \phi_m^{(0)} \middle| \delta H \middle| \phi_0^{(0)} \right\rangle}{E_0^{(0)} - E_m^{(0)}} \left|\phi_m^{(0)}\right\rangle,$

$$E_0^{(1)} = \left\langle \phi_0^{(0)} \middle| \delta H_{\text{int}} \middle| \phi_0^{(0)} \right\rangle = \frac{1}{2} N g n$$

(Hartree energy)

Second-order perturbation

$$E_0^{(2)} = \sum_{m\neq 0} \frac{1}{E_0^{(0)} - E_m^{(0)}} \left\langle \phi_0^{(0)} \middle| \delta H \middle| \phi_m^{(0)} \right\rangle \left\langle \phi_m^{(0)} \middle| \delta H \middle| \phi_0^{(0)} \right\rangle,$$

Collision process

$$\mathbf{k}_1 \uparrow \rightarrow \mathbf{k}_1 + \mathbf{Q} \uparrow \rightarrow \mathbf{k}_1 \uparrow, \quad (k_1 < k_F, \ |\mathbf{k}_1 + \mathbf{Q}| > k_F)$$

$$\mathbf{k}_2 \downarrow \rightarrow \mathbf{k}_2 - \mathbf{Q} \downarrow \rightarrow \mathbf{k}_2 \downarrow, \quad (k_2 < k_F, \ |\mathbf{k}_2 - \mathbf{Q}| > k_F)$$

$$\left| \phi_m^{(0)} \right\rangle = \psi^+_{\mathbf{k}_1+\mathbf{Q}\uparrow} \psi^+_{\mathbf{k}_2-\mathbf{Q}\downarrow} \psi_{\mathbf{k}_2\downarrow} \psi_{\mathbf{k}_1\uparrow} \left| \phi_0^{(0)} \right\rangle,$$

$$E_0^{(2)} = \frac{g^2}{V^2} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{Q}} \frac{n_{\mathbf{k}_1} n_{\mathbf{k}_2} (1 - n_{\mathbf{k}_1+\mathbf{Q}})(1 - n_{\mathbf{k}_2-\mathbf{Q}})}{\varepsilon_{\mathbf{k}_1} + \varepsilon_{\mathbf{k}_2} - \varepsilon_{\mathbf{k}_1+\mathbf{Q}} - \varepsilon_{\mathbf{k}_2-\mathbf{Q}}}, \qquad n_k = \theta(k_F - k)$$

Ground state energy $E_0 = E_0^{(0)} + E_0^{(1)} + E_0^{(2)}$

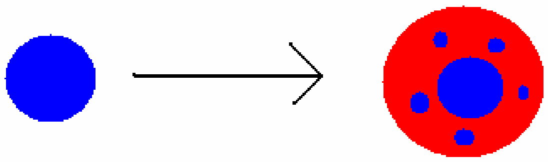
$$E_0 = N\varepsilon_F \left[\frac{3}{5} + \frac{2}{3\pi} k_F a + \frac{4}{35\pi^2}(11 - 2\ln 2)(k_F a)^2\right]$$

(Valid when $k_F a$ <1)

4. Fermi Liquid Theory

Adiabatic-continuation principle:

By gradually turning on the interaction, an eigenstate of an ideal Fermi gas can evolve smoothly into an eigenstate of an interacting Fermi gas. Thus the excitation structure are the same for interacting and non-interacting Fermi gases.



Ideal Fermi gas

Fermi Liquid

Fermi atom

Dressed quasiparticle

Landau’s assumptions:

(1) There is one-to-one correspondence between excitations in an ideal Fermi gas and excitations in a Fermi Liquid. Fermi-liquid state can be described by the quasi-particle occupation number $n_{\mathbf{k}}$, $n_{\mathbf{k}} = \theta(k_F - k)$ for ground state.

(2) Quasi-particle energies are determined by the interaction parameter $f(\mathbf{k},\mathbf{k}')$ and occupation number $n_{\mathbf{k}}$.

$$\delta\varepsilon_{\mathbf{k}} = \int \frac{d^3k'}{(2\pi)^3} f(\mathbf{k},\mathbf{k}')\delta n_{\mathbf{k}'}$$

Effective mass of quasi-particles

$$\frac{1}{m^*} = \frac{1}{m} - \frac{k_F}{4\pi^2\hbar^2}\int_0^\pi f(\theta)\cos(\theta)d\theta$$

where $\cos(\theta) = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}', \; f(\theta) \equiv f(k_F\hat{\mathbf{k}}, k_F\hat{\mathbf{k}}')$

Current of quasi-particles is the same as current of particles,

$$\int \frac{d^3k}{(2\pi)^3}\left(\frac{1}{\hbar}\nabla_{\mathbf{k}}\varepsilon_{\mathbf{k}}\right)n_{\mathbf{k}}$$

Total momentum of quasi-particles is the same as total momentum of particles,

$$\int \frac{d^3k}{(2\pi)^3}\hbar\mathbf{k}n_{\mathbf{k}} = m\int \frac{d^3k}{(2\pi)^3}\left(\frac{1}{\hbar}\nabla_{\mathbf{k}}\varepsilon_{\mathbf{k}}\right)n_{\mathbf{k}},$$

$$\int \frac{d^3k}{(2\pi)^3}\hbar\mathbf{k}\delta n_{\mathbf{k}} = m\int \frac{d^3k}{(2\pi)^3}\left(\frac{1}{\hbar}\nabla_{\mathbf{k}}\varepsilon_{\mathbf{k}}\right)\delta n_{\mathbf{k}} + m\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\left[\frac{1}{\hbar}\nabla_{\mathbf{k}} f(\mathbf{k},\mathbf{k}')\right]n_{\mathbf{k}}\delta n_{\mathbf{k}'},$$

$$\frac{\hbar\mathbf{k}}{m} = \frac{\hbar k_F}{m^*}\hat{\mathbf{k}} - \int \frac{d^3k'}{(2\pi)^3}\frac{1}{\hbar} f(\mathbf{k}',\mathbf{k})\nabla_{\mathbf{k}'}n_{\mathbf{k}'},$$

$$\frac{1}{m} = \frac{1}{m^*} + \int_0^\pi \frac{1}{4\pi^2\hbar^2} f(\theta)\cos(\theta)d\theta$$

(3) Quasi-particle distribution satisfies Boltzmann equation in the long-wavelength and low-frequency limit.

$$\hbar\frac{\partial n}{\partial t}+\nabla_{\mathbf{k}}\varepsilon\cdot\nabla_{\mathbf{r}}n-\nabla_{\mathbf{r}}\varepsilon\cdot\nabla_{\mathbf{k}}n=I(n),$$

where $n\equiv n_{\mathbf{k}}(\mathbf{r},t),\quad \varepsilon\equiv\varepsilon_{\mathbf{k}}(\mathbf{r},t)$

Consider small fluctuation without collision

$$\hbar\frac{\partial\delta n}{\partial t}+\nabla_{\mathbf{k}}\varepsilon\cdot\nabla_{\mathbf{r}}\delta n-\nabla_{\mathbf{k}}n\cdot\nabla_{\mathbf{r}}\int\frac{d^3k'}{(2\pi)^3}f(\mathbf{k},\mathbf{k}')]\delta n_{\mathbf{k}'}=0,$$

Assume $\delta n\equiv\delta(\varepsilon_{\mathbf{k}}-\varepsilon_F)g(\hat{\mathbf{k}})\exp[i(\mathbf{q}\cdot\mathbf{r}-\omega t)],$

$$(\omega-v_F\hat{\mathbf{k}}\cdot\mathbf{q})g(\hat{\mathbf{k}})=\hat{\mathbf{k}}\cdot\mathbf{q}\frac{k_F^2}{4\pi^3\hbar}\int f(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')g(\hat{\mathbf{k}}')d\hat{\mathbf{k}}',$$

For small and constant f, $\omega\approx v_F q$

Zero sound in the collisionless regime $v\approx v_F,\quad \omega\tau\gg 1;$

Ordinary sound in the hydrodynamical regime $v\approx v_F/\sqrt{3},\quad \omega\tau\ll 1$

Factorization of the interaction parameter,

$$f_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}(\theta) = \sum_l (f_{1l} + \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' f_{2l}) P_l[\cos(\theta)]$$

The interaction parameter for a dilute Fermi gas can be obtained from perturbation theory,

$$f_{1l} = \frac{g}{2}[\delta_{l,0} + O(k_F a)], \quad f_{2l} = -\frac{g}{2}[\delta_{l,0} + O(k_F a)],$$

with the effective mass $\frac{m^*}{m} = 1 + \frac{8}{15\pi^2}(7\ln 2 - 1)(k_F a)^2.$

Quadrupole modes in a trapped collisionless Fermi gas

$$\omega = 2\Omega_\perp, \quad \omega = 2\Omega_z$$

L. Vichi, J. Low Temp. Phys. 121, 177 (2000).