

NONRELATIVISTIC EFFECTIVE FIELD THEORIES

AND

RENORMALONS

ANTONIO PINEDA

(IFAE, Universitat Autònoma de Barcelona)

Question

$$M_{\Upsilon(1S)} = m_{\text{OS}}(1 + A_2\alpha_s^2 + A_3\alpha_s^3 + \dots)$$

What if $A_n \sim n!$? Bad convergence

Should we expect that?

Renormalons

They are a potential problem in effective field theories of QCD (OPE) where the matching coefficients can be computed in perturbation theory.

Examples:

OPE

NRQCD

HQET

pNRQCD

SCET

Renormalons appear as soon as we have **factorization** between different scales: They can deteriorate the convergence of the perturbative series in **QCD**.

Can one understand the **renormalon** within an **effective field theory/factorization** formalism?

Problems:

- 1) Fix the parameters of the **Standard Model**. Search for weakly sensitive to long distance physics observables. One wants to avoid spurious dependence on the renormalon.
- 2) **Meaningful** determination of non-perturbative parameters.

$$\mathcal{L} = \sum_n \frac{1}{m^n} c_n O_n$$

Matching coefficients suffer from renormalon ambiguities that cancel with the ones of the matrix elements in effective field theory calculations.

$$c(\nu) = \bar{c} + \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}.$$

Its Borel transform would be

$$B[c](t) \equiv \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and c is written in terms of its Borel transform as

$$c = \bar{c} + \int_0^{\infty} dt e^{-t/\alpha_s} B[c](t).$$

The ambiguities in the matching coefficient ($c_n \sim n!$) reflects in poles in the Borel transform. If we take the one closest to the origin,

$$\delta B[c](t) \sim \frac{1}{a - t},$$

where a is a positive number, it sets up the maximal accuracy with which one can obtain the matching coefficients from a perturbative calculation, which is (roughly) of the order of

$$\delta c \sim r_{n^*} \alpha_s^{n^*},$$

where $n^* \sim \frac{a}{\alpha_s}$. Moreover, the fact that a is positive means that, even after Borel resummation, c suffers from a non-perturbative ambiguity of order

$$\delta c \sim (\Lambda_{QCD})^{\frac{a\beta_0}{2\pi}}.$$

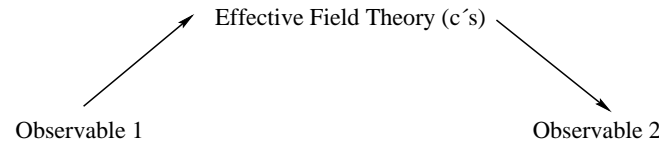


Figure 1: Symbolic relation between observables through the determination of the matching coefficients of the effective field theory.

Examples

$$M_B = m_{OS} + \bar{\Lambda} + \mathcal{O}(1/m_{OS})$$

M_B is renormalon free. **Problem.** Therefore m_{OS} suffers from renormalon ambiguities:

$$m_{OS} = m_{\overline{MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \dots)$$

with $B_n \sim n!$. In other words

$$\delta_{np}^{(\text{pert.})} m_{OS} = \delta_{np}^{(\text{pert.})} m_{\overline{MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \dots) \sim \Lambda_{QCD}!$$

On the other hand

$$M_{\Upsilon(1S)} = m_{OS}(1 + A_2\alpha_s^2 + A_3\alpha_s^3 + \dots) + \mathcal{O}\left(\frac{\Lambda_{QCD}^3}{(m_{OS}\alpha_s)^2}\right).$$

$M_{\Upsilon(1S)}$ is renormalon free. Therefore, the perturbative series suffers from renormalon ambiguities: $A_n \sim n!$

$$\delta_{np}^{(\text{pert.})} M_{\Upsilon(1S)} = \delta_{np}^{(\text{pert.})} m_{OS}(1 + A_2\alpha_s^2 + A_3\alpha_s^3 + \dots) \sim \Lambda_{QCD}.$$

Physics. Computations to **n-loops** produce small scales: me^{-n} . From the effective field theory point of view these scales should be in the effective field theory instead that in the matching coefficients.

Proposal: to subtract the renormalon from the matching coefficients.

OS mass

$$m_{\text{OS}} = m_{\overline{\text{MS}}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1},$$

The behavior of the perturbative expansion at large orders is dictated by the closest singularity to the origin of its Borel transform ($u = \frac{\beta_0 t}{4\pi}$).

$$B[m_{\text{OS}}](t(u)) = N_m \nu \frac{1}{(1-2u)^{1+b}} (1 + c_1(1-2u) + c_2(1-2u)^2 + \dots) + (\text{analytic term}),$$

Next renormalon at $u = 1$.

$$r_n \stackrel{n \rightarrow \infty}{\sim} N_m \nu \left(\frac{\beta_0}{2\pi} \right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \dots \right).$$

$$b = \frac{\beta_1}{2\beta_0^2}, \quad c_1 = \frac{1}{4b\beta_0^3} \left(\frac{\beta_1^2}{\beta_0} - \beta_2 \right), \quad \dots$$

Determination of N_m

$$\begin{aligned} D_m(u) &= \sum_{n=0}^{\infty} D_m^{(n)} u^n = (1-2u)^{1+b} B[m_{\text{OS}}](t(u)) \\ &= N_m \nu (1 + c_1(1-2u) + c_2(1-2u)^2 + \dots) + (1-2u)^{1+b} (\text{analytic term}). \end{aligned}$$

$$N_m \nu = D_m(u = 1/2).$$

$$\begin{aligned} N_m &= 0.4244 + 0.1379 + 0.0127 = 0.5750 \quad (n_f = 3) \\ &= 0.4244 + 0.1275 + 0.0004 = 0.5523 \quad (n_f = 4) \\ &= 0.4244 + 0.1199 - 0.0208 = 0.5235 \quad (n_f = 5) \end{aligned}$$

$$\nu \sim m$$

Large β_0 analysis

$$m \left(\frac{\nu}{m} \right)^{2u} \simeq \nu \left\{ 1 + (2u - 1) \ln \frac{\nu}{m} + \dots \right\}.$$

Therefore, the underlying assumption is that we are in a regime where (besides $2u - 1 \ll 1$)

$$(2u - 1) \ln \frac{\nu}{m} \ll 1.$$

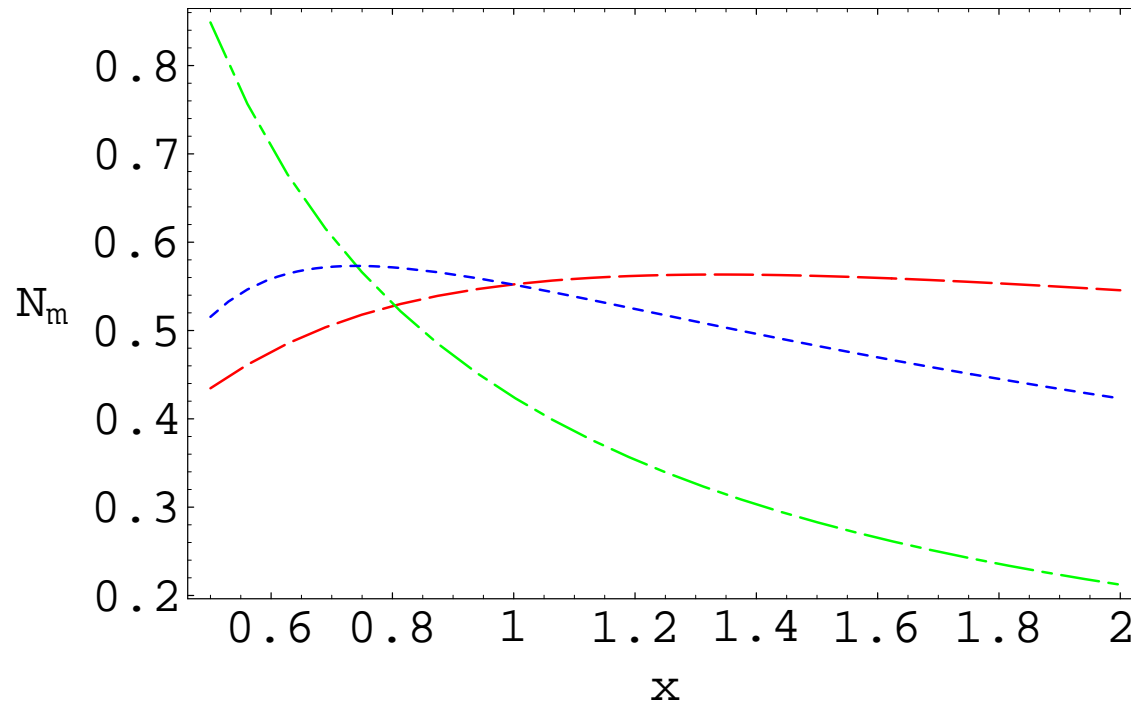


Figure 2: $x \equiv \frac{\nu}{m_{\text{MS}}}$ dependence of N_m for $n_f = 4$.

Estimates of r_n

$\tilde{r}_n = r_n/m_{\overline{\text{MS}}}$	\tilde{r}_0	\tilde{r}_1	\tilde{r}_2	\tilde{r}_3	\tilde{r}_4
exact ($n_f = 3$)	0.424413	1.04556	3.75086	---	---
our estimate ($n_f = 3$)	0.617148	0.977493	3.76832	18.6697	118.441
large β_0 ($n_f = 3$)	0.424413	1.42442	3.83641	17.1286	97.5872
exact ($n_f = 4$)	0.424413	0.940051	3.03854	---	---
our estimate ($n_f = 4$)	0.645181	0.848362	3.03913	13.8151	80.5776
large β_0 ($n_f = 4$)	0.424413	1.31891	3.28911	13.5972	71.7295
exact ($n_f = 5$)	0.424413	0.834538	2.36832	---	---
our estimate ($n_f = 5$)	0.706913	0.713994	2.36440	9.73117	51.5952
large β_0 ($n_f = 5$)	0.424413	1.21339	2.78390	10.5880	51.3865

Table 1: Values of r_n for $\nu = m_{\overline{\text{MS}}}$. Either the exact result (when available), our estimate, or the estimate using the large β_0 approximation.

$\tilde{r}_n = r_n/m_{\overline{\text{MS}}}$	\tilde{r}_0	\tilde{r}_1	\tilde{r}_2	\tilde{r}_3	\tilde{r}_4
$O(1/n)$ ($n_f = 3$)	-0.164	-0.046	-0.027	-0.019	-0.015
$O(1/n^2)$ ($n_f = 3$)	0.237	-0.103	-0.017	-0.007	-0.004
$O(1/n)$ ($n_f = 4$)	-0.105	-0.028	-0.016	-0.012	-0.009
$O(1/n^2)$ ($n_f = 4$)	0.274	-0.126	-0.020	-0.008	-0.004
$O(1/n)$ ($n_f = 5$)	0.024	0.006	0.003	0.002	0.002
$O(1/n^2)$ ($n_f = 5$)	0.326	-0.165	-0.023	-0.009	-0.005

Table 2: $O(1/n)$ corrections (normalized with respect the leading solution) of our r_n estimates for different number of light fermions.

$$c_1(n_f = 0) \simeq -0.215, \quad c_2(n_f = 0) \simeq 0.185$$

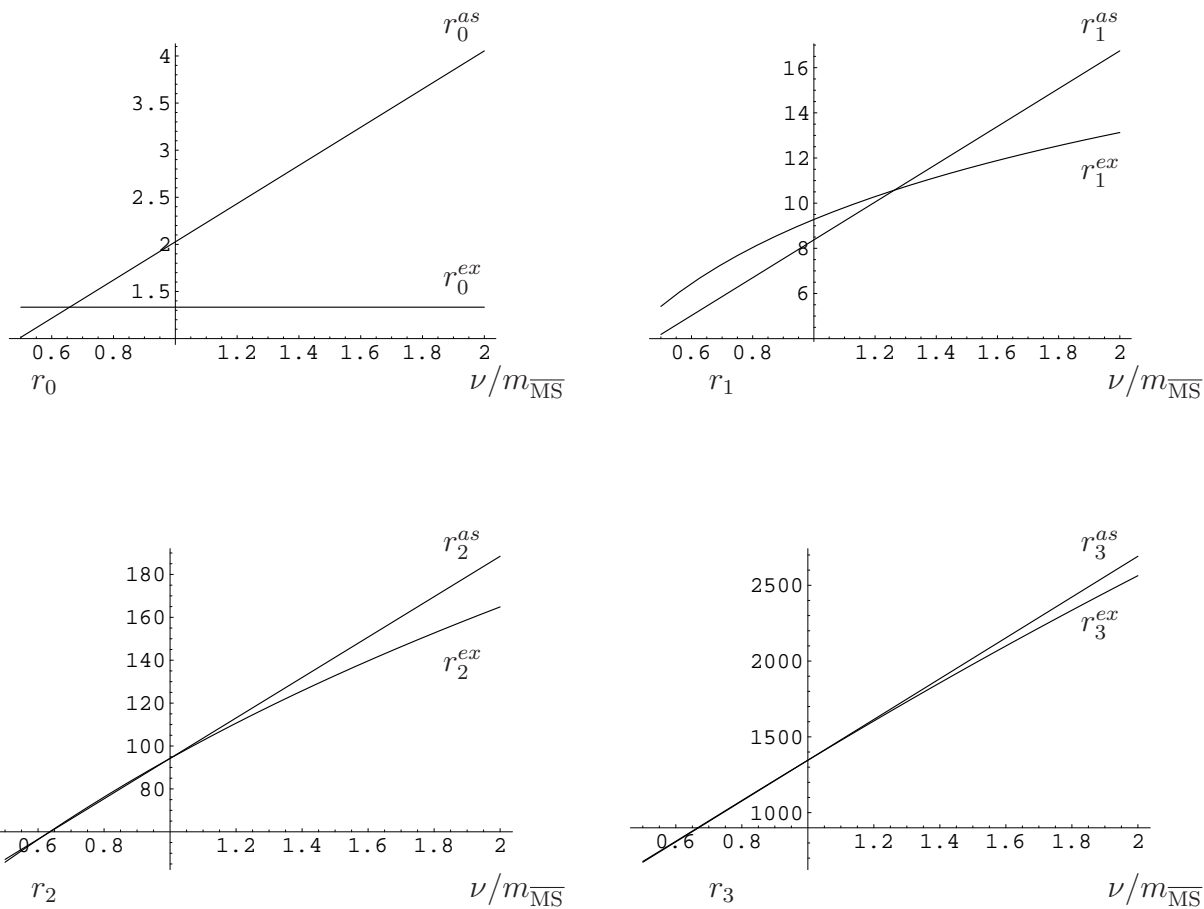


Figure 3: Plots of the exact (r_n^{ex}) and asymptotic (r_n^{as}) value of $r_n(\nu)$ at different orders in perturbation theory as a function of $\nu/m_{\overline{\text{MS}}}$. The scale dependence of r_3^{ex} is known exactly. The constant term has been fixed using renormalon dominance.

$$r_n \stackrel{n \rightarrow \infty}{\sim} m_{\overline{\text{MS}}} \left(\frac{\beta_0}{2\pi} \right)^n n! N_m \sum_{s=0}^n \frac{\ln^s[\nu/m_{\overline{\text{MS}}}]}{s!},$$

$$r_n \stackrel{n \rightarrow \infty}{\equiv} N_m \nu \left(\frac{\beta_0}{2\pi} \right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \dots \right).$$

Renormalon subtracted matching and power counting

Effective field theory with renormalon free parameters but preserving the power counting rules.

The renormalon is associated to the non-analytic behavior in $1 - 2u$. These terms also exist in the effective theory. **Procedure:** to explicitly subtract them from the matching coefficients (the mass).

$$B[m_{\text{RS}}] \equiv B[m_{\text{OS}}] - N_m \nu_f \frac{1}{(1 - 2u)^{1+b}} (1 + c_1(1 - 2u) + c_2(1 - 2u)^2 + \dots),$$

$$m_{\text{RS}}(\nu_f) = m_{\text{OS}} - \sum_{n=0}^{\infty} N_m \nu_f \left(\frac{\beta_0}{2\pi} \right)^n \alpha_s^{n+1}(\nu_f) \sum_{k=0}^{\infty} c_k \frac{\Gamma(n + 1 + b - k)}{\Gamma(1 + b - k)}.$$

Expansion in $\alpha_s(\nu)$

$$m_{\text{RS}}(\nu_f) = m_{\overline{\text{MS}}} + \sum_{n=0}^{\infty} r_n^{\text{RS}} \alpha_s^{n+1},$$

where $r_n^{\text{RS}} = r_n^{\text{RS}}(m_{\overline{\text{MS}}}, \nu, \nu_f)$. They are the ones expected to be of natural size. We now do not lose accuracy if we first obtain m_{RS} and later on $m_{\overline{\text{MS}}}$.

Different scheme

$$B[m_{\text{RS}'}] \equiv B[m_{\text{RS}}] + N_m \nu_f (1 + c_1 + c_2 + \dots).$$

Check of convergence improvement

Masses	$O(\alpha_s)$	$O(\alpha_s^2)$	$O(\alpha_s^3)$	$O(\alpha_s^4)$	total
m_{OS}	401	199	144	147	5 102
m_{RS}	111	50	17	7	4 395
$m_{RS'}$	401	114	38	15	4.778
m_{PS}	210	80	42	---	4 542
$m_{1S}^{(static)}$	102	50	19	8	4 389
m_{RS}	256	95	40	21	4 622
$m_{RS'}$	401	157	74	41	4.882
m_{PS}	306	120	67	---	4.703
$m_{1S}^{(static)}$	251	94	41	22	4 619

Table 3: Contributions at various orders in α_s for different mass definitions for the bottom quark case, either with $\nu_f = 1/r = 2$ GeV (middle panel) or with $\nu_f = 1/r = 1$ GeV (lower panel). The results are displayed in MeV. For the $O(\alpha_s^4)$ results, the estimate from Table 1 has been used. The other parameters have been fixed to the values $m_{\overline{MS}}(m_{\overline{MS}}) = 4.21$ GeV, $\nu = m_{\overline{MS}}(m_{\overline{MS}})$ and $n_f = 4$.

$$m_{1S}^{(static)} \equiv m_{OS} + \frac{V(r)}{2} = m_{\overline{MS}} + \left(r_0 - \frac{C_f}{2r} \right) \alpha_s + \dots$$

HQET

$$\mathcal{L} = \bar{h} (iD_0 - \delta m_{RS}) h + O\left(\frac{1}{m_{RS}}\right),$$

where $\delta m_{RS} = m_{OS} - m_{RS}$ and similarly for the NRQCD Lagrangian.

Weakly sensitive to long distance physics observable

$$\langle M_B \rangle - \langle M_D \rangle = m_{b,RS} - m_{c,RS} + \lambda_1 \left(\frac{1}{2m_{b,RS}} - \frac{1}{2m_{c,RS}} \right) + O(1/m_{RS}^2).$$

pNRQCD. If $\Lambda_{QCD} \ll m\alpha_s$

$$V_{s,RS(RS')}^{(0)}(\nu_f) = V_s^{(0)} + 2\delta m_{RS(RS')},$$

Check of convergence improvement

Potentials	$O(\alpha_s)$	$O(\alpha_s^2)$	$O(\alpha_s^3)$	$O(\alpha_s^4)$	total
$V_s^{(0)}$	-910	-306	-302	-383	-1 902
$V_{s,RS}^{(0)}$	-205	3	-2	-3	-208
$V_{s,RS'}^{(0)}$	-910	-54	-14	-6	-984
$V_{s,PS}^{(0)}$	-446	-42	-25	---	-513
$V_{s,RS}^{(0)}$	-558	-63	-41	-26	-687
$V_{s,RS'}^{(0)}$	-910	-180	-95	-54	-1 239
$V_{s,PS}^{(0)}$	-678	-116	-75	---	-869

Table 4: Contributions at various orders in α_s for different singlet static potential definitions for some typical scales in the Υ system, either with $\nu_f = 2$ GeV (middle panel) or with $\nu_f = 1$ GeV (lower panel). The results are displayed in MeV. For the $O(\alpha_s^4)$ results, the estimate from Table 5 has been used. The other parameters have been fixed to the values $\nu = 1/r = 2.5$ GeV and $n_f = 4$.

pNRQCD Lagrangian

$$\begin{aligned} \mathcal{L}^{(0)} = & \text{Tr} \left\{ S^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m_{RS}} + \sum_n \frac{V_{s,RS}^{(n)}(\mathbf{x})}{m_{RS}^n} \right) S + O^\dagger \left(iD_0 - \frac{\mathbf{p}^2}{m_{RS}} + \sum_n \frac{V_{o,RS}^{(n)}(\mathbf{x})}{m_{RS}^n} \right) O \right\} \\ & + gV_A(\mathbf{x}) \text{Tr} \left\{ O^\dagger \mathbf{x} \cdot \mathbf{E} S + S^\dagger \mathbf{x} \cdot \mathbf{E} O \right\} + g \frac{V_B(\mathbf{x})}{2} \text{Tr} \left\{ O^\dagger \mathbf{x} \cdot \mathbf{E} O + O^\dagger O \mathbf{x} \cdot \mathbf{E} \right\}, \end{aligned}$$

Weakly sensitive to long distance physics observable

$$M_{nlj} = 2m_{RS} + \sum_{m=2}^{\infty} A_{nlj}^{m,RS}(\nu_{us}) \alpha_s^m + \delta M_{nlj}^{US}(\nu_{us}).$$

The static potential

$$V_s^{(0)}(r; \nu_{us}) = \sum_{n=0}^{\infty} V_{s,n}^{(0)} \alpha_s^{n+1},$$

$2m_{OS} + V_s^{(0)}$ (not $2m_{OS} + V_o^{(0)}$) can be understood as an observable up to $O(r^2 \Lambda_{QCD}^3, \Lambda_{QCD}^2/m)$ renormalon (and/or non-perturbative) contributions. We can use our knowledge of the asymptotic behavior of m_{OS} .

$$V_{s,n}^{(0)} \stackrel{n \rightarrow \infty}{\equiv} N_V \nu \left(\frac{\beta_0}{2\pi} \right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \dots \right)$$

$$2N_m + N_V = 0$$

$$\begin{aligned} D_V(u) &= \sum_{n=0}^{\infty} D_V^{(n)} u^n = (1-2u)^{1+b} B[V_s^{(0)}](t(u)) \\ &= N_V \nu (1 + c_1(1-2u) + c_2(1-2u)^2 + \dots) + (1-2u)^{1+b} (\text{analytic term}). \end{aligned}$$

Next (IR) renormalon at $u = 3/2$.

$$\begin{aligned} N_V &= -1.333 + 0.572 - 0.345 = -1.107 \quad (n_f = 3) \\ &= -1.333 + 0.585 - 0.329 = -1.077 \quad (n_f = 4) \\ &= -1.333 + 0.587 - 0.295 = -1.042 \quad (n_f = 5). \end{aligned}$$

$$2 \frac{2N_m + N_V}{2N_m - N_V} = \begin{cases} 0.038 & , n_f = 3 \\ 0.025 & , n_f = 4 \\ 0.005 & , n_f = 5. \end{cases}$$

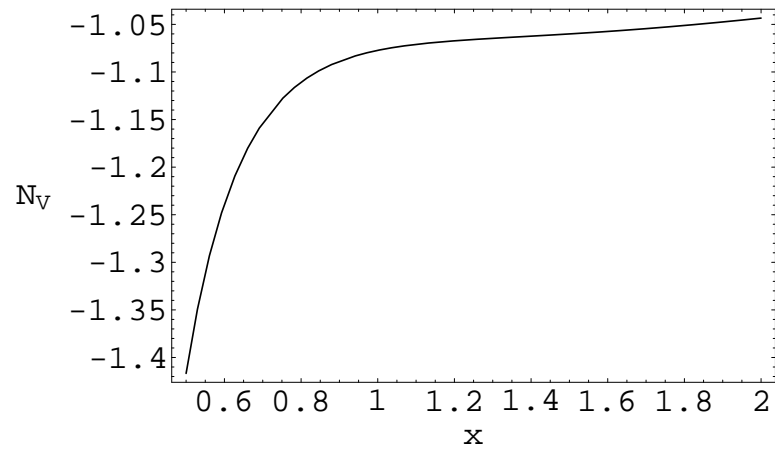
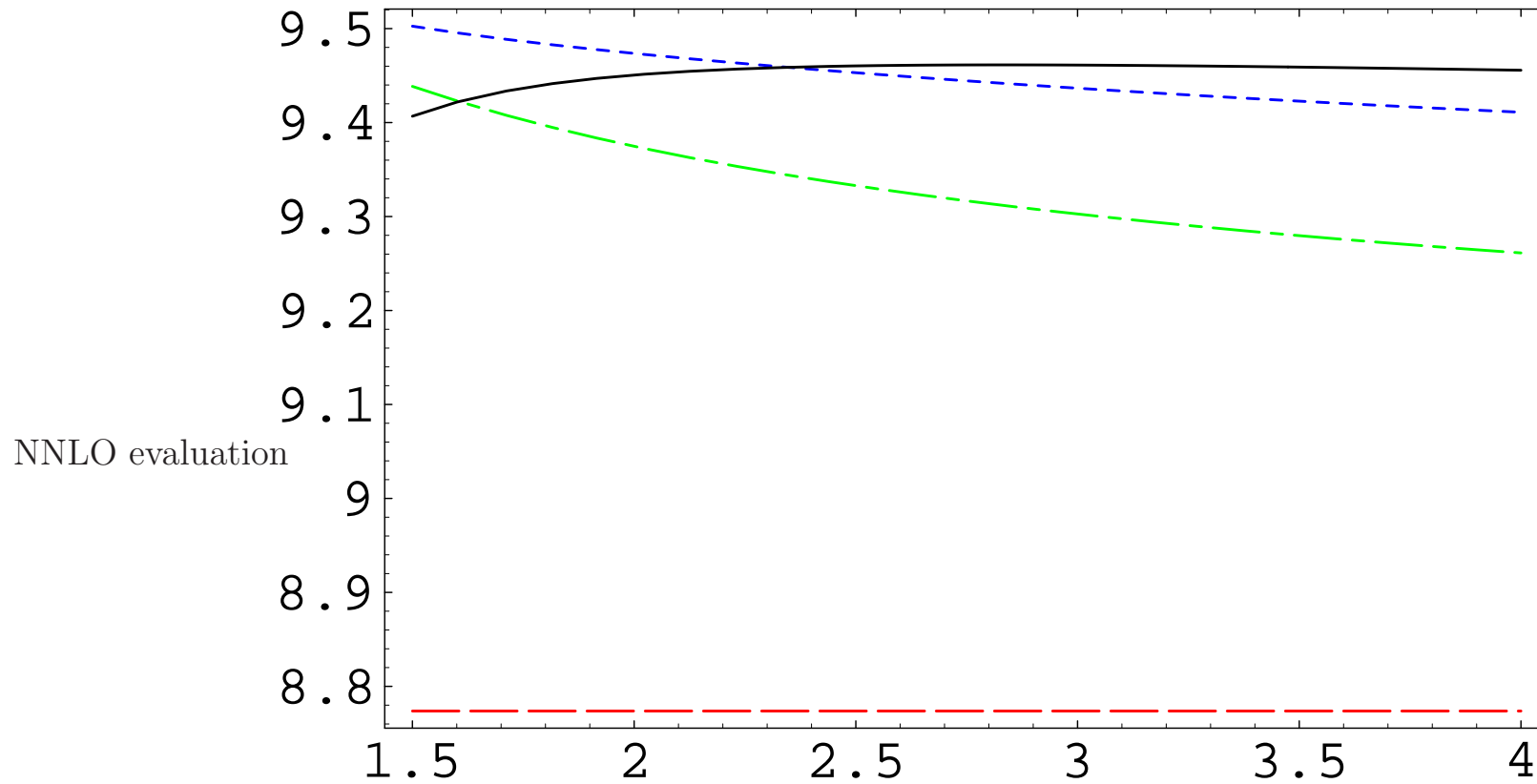


Figure 4: $x \equiv \nu r$ dependence of N_V for $n_f = 4$.

$\tilde{V}_{s,n}^{(0)} = rV_{s,n}^{(0)}$	$\tilde{V}_{s,0}^{(0)}$	$\tilde{V}_{s,1}^{(0)}$	$\tilde{V}_{s,2}^{(0)}$	$\tilde{V}_{s,3}^{(0)}$	$\tilde{V}_{s,4}^{(0)}$
exact ($n_f = 3$)	-1.33333	-1.84512	-7.28304	---	---
our estimate ($n_f = 3$)	-1.23430	-1.95499	-7.53665	-37.3395	-236.882
large β_0 ($n_f = 3$)	-1.33333	-2.69395	-7.69303	-34.0562	---
exact ($n_f = 4$)	-1.33333	-1.64557	-5.94978	---	---
our estimate ($n_f = 4$)	-1.29036	-1.69672	-6.07826	-27.6301	-161.155
large β_0 ($n_f = 4$)	-1.33333	-2.49440	-6.59553	-27.0349	---
exact ($n_f = 5$)	-1.33333	-1.44602	-4.70095	---	---
our estimate ($n_f = 5$)	-1.41383	-1.42799	-4.72881	-19.4623	-103.190
large β_0 ($n_f = 5$)	-1.33333	-2.29485	-5.58246	-21.0518	---

Table 5: Values of $V_{s,n}^{(0)}$ with $\nu = 1/r$. Either the exact result (when available), our estimate, or the estimate using the large β_0 approximation.

Bottom $\overline{\text{MS}}$ quark mass determination



Dependence on the parameters for the **RS** scheme: $\nu = 2.5_{-1}^{+1.5}$ GeV, $\nu_f = 2 \pm 1$ GeV, $\alpha_s(M_z) = 0.118 \pm 0.003$ and $N_m = 0.552 \pm 0.0552$

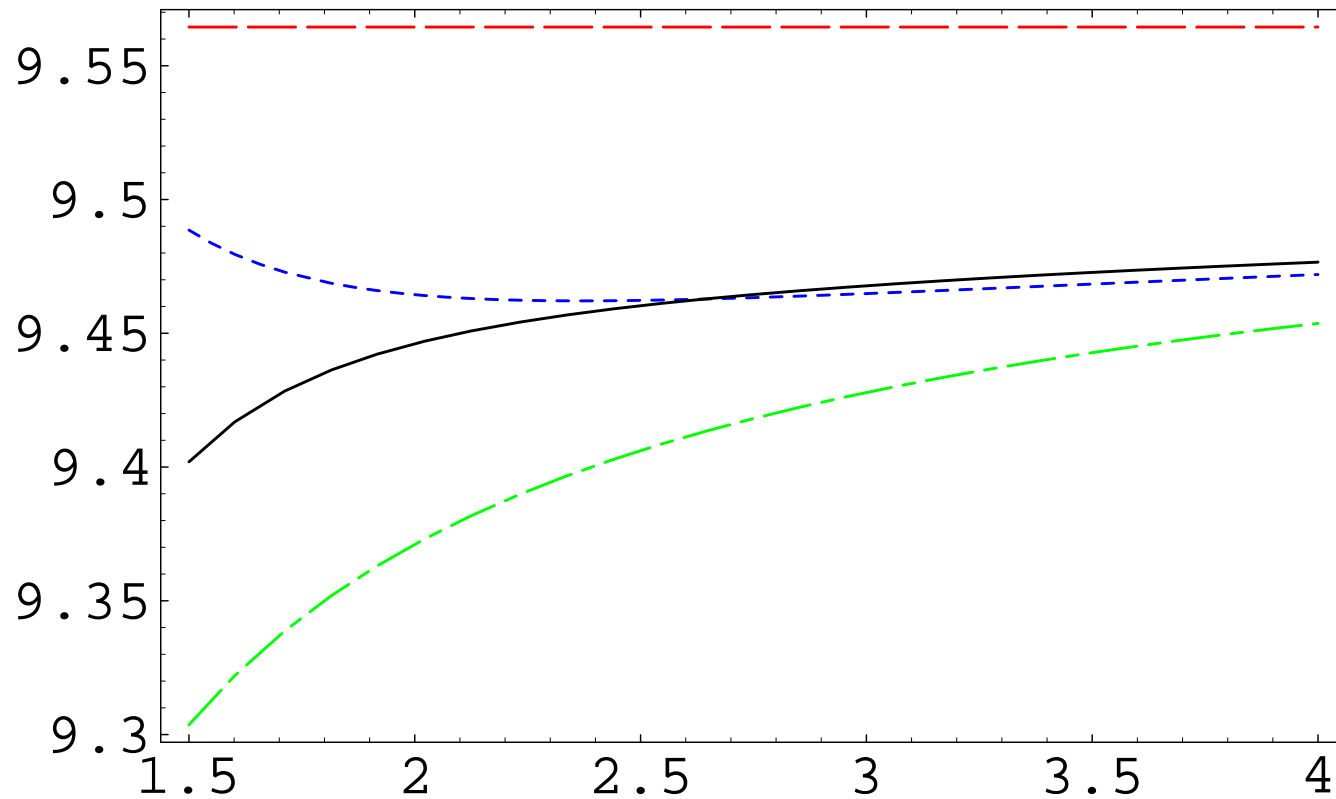
$$m_{b,\text{RS}}(2 \text{ GeV}) = 4387_{+28}^{+2} (\nu)_{+7}^{-5} (\nu_f)_{+16}^{-16} (\alpha_s)_{+68}^{-68} (N_m) \text{ MeV};$$

$$m_{b,\overline{\text{MS}}}(m_{b,\overline{\text{MS}}}) = 4203_{+25}^{+2} (\nu)_{+6}^{-5} (\nu_f)_{+27}^{-28} (\alpha_s)_{+10}^{-10} (N_m) \text{ MeV}.$$

Convergence. In the **RS** scheme

$$M_{\Upsilon(1S)} = 8774 + 559 + 120 + 7 \text{ MeV}.$$

NNLO(st. pot.) $\sim +62$ MeV. NNLO(rel.) ~ -55 MeV.



For the **RS'** scheme, we obtain the result

$$m_{b,RS'}(2 \text{ GeV}) = 4\,782_{+31}^{-08}(\nu)_{+3}^{-7}(\nu_f)_{-12}^{+15}(\alpha_s)_{+28}^{-28}(N_m) \text{ MeV};$$

$$m_{b,\overline{\text{MS}}}(m_{b,\overline{\text{MS}}}) = 4\,214_{+28}^{-08}(\nu)_{+3}^{-6}(\nu_f)_{+25}^{-25}(\alpha_s)_{+9}^{-9}(N_m) \text{ MeV}.$$

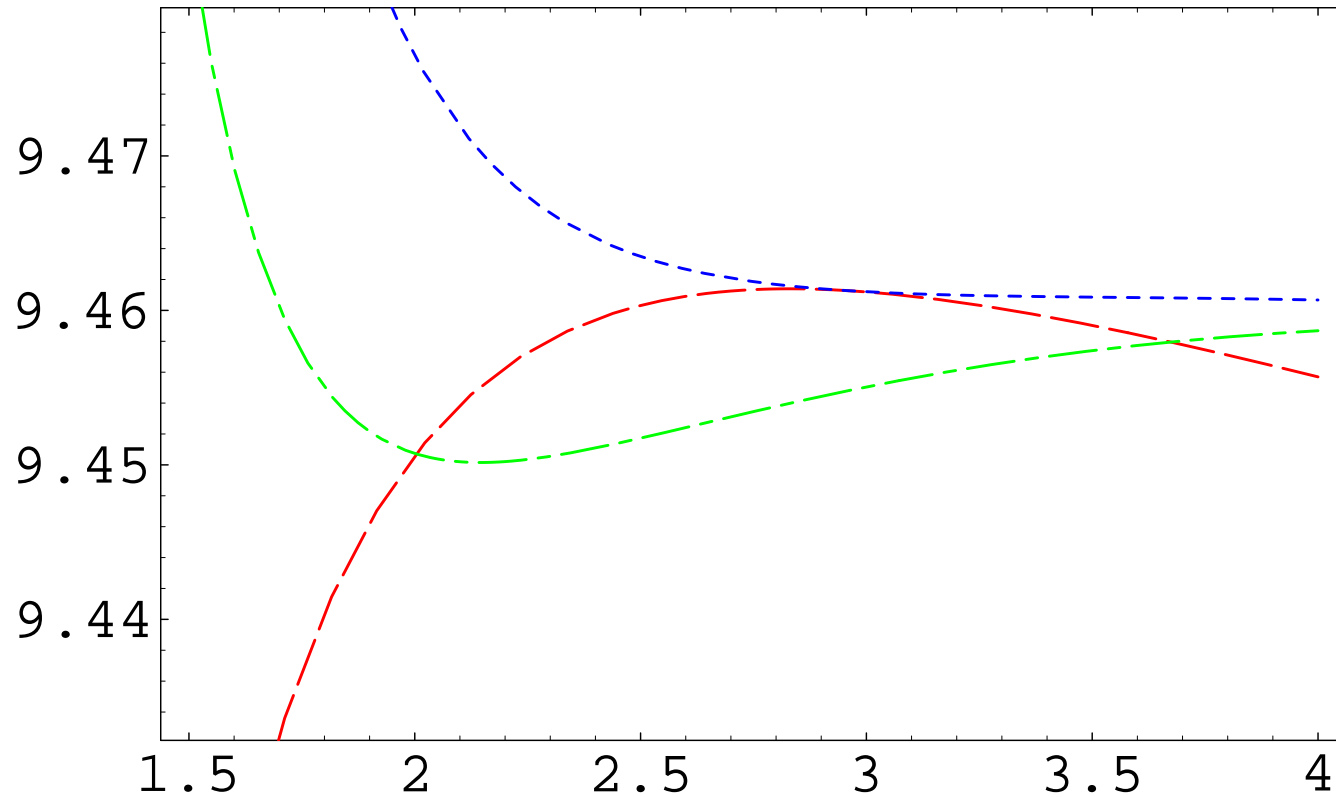
Convergence. In the **RS'** scheme

$$M_{\Upsilon(1S)} = 9\,564 - 158 + 56 - 2 \text{ MeV}.$$

NNLO(st. pot.) $\sim +45$ MeV. NNLO(rel.) ~ -47 MeV.

NNLO evaluation. Exact scale dependence + large β_0 estimate for the log-independent piece of $A_{101}^{5,OS}$.

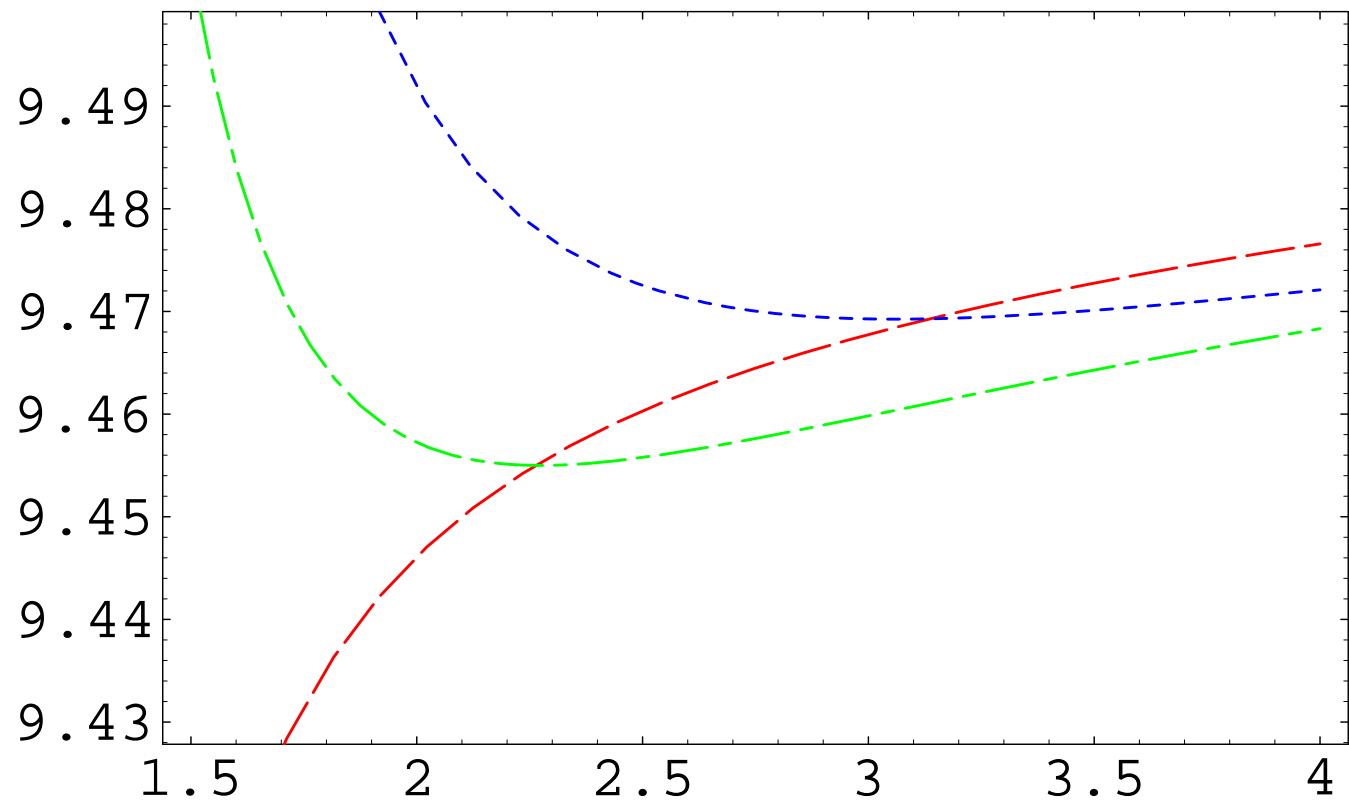
Two kind of logs: (a) $\sim \ln(m\alpha_s/\nu)$ and (b) hard/ultrasoft logs $\sim \ln(m), \ln(\nu_{us})$.



Strong scale dependence at small ν (renormalization group improvement needed?)

NNLO contribution seems to be under control.

Error $\sim \pm 50$ MeV (including charm mass effects).



What about **US** effects?

$$\delta M_{nlj}^{\text{US}}(\nu_{us}) \simeq \frac{T_F}{3N_c} \int_0^\infty dt \langle n, l | \mathbf{r} e^{-t(H_o^{\text{RS}} - E_n^{\text{RS}})} \mathbf{r} | n, l \rangle$$

$$\times \langle g\mathbf{E}^a(t) \phi(t, 0)_{ab}^{\text{adj}} g\mathbf{E}^b(0) \rangle(\nu_{us}),$$

where $H_o^{\text{RS}} \equiv \frac{\mathbf{p}^2}{m_{\text{RS}}} + \frac{1}{2N_c} \frac{\alpha_s}{r}$ and $E_n^{\text{RS}} \equiv -m_{\text{RS}} C_f^2 \alpha_s^2 / (4n^2)$.

Situation $\Lambda_{\text{QCD}} \ll m\alpha_s^2$

$$\delta M_{nl}^{\text{US}}(\nu_{us}) = \delta M_{nl}^{\text{US, pert.}} + \delta M_{nl}^{\text{US, no-pert.}},$$

$$\delta M_{nl}^{\text{US, pert.}} \sim m_{\text{RS}} \alpha_s^5 \ln \frac{\nu_{us}}{m_{\text{RS}} \alpha_s},$$

$$\delta M_{nl}^{\text{US, no-pert.}} = \sum_{n=0}^{\infty} C_n O_n,$$

where $C_n \sim 1/(m_{\text{RS}}^{3+2n} \alpha_s^{4+4n})$ and $O_n \sim \Lambda_{\text{QCD}}^{4+2n}$.

Problems: (1) Dependence on $\alpha_s(m\alpha_s^2)$. $\langle H_o - E_1 \rangle_{10} \sim 360 \text{ MeV}$ (up to numerical factors). (2) Convergence of the OPE (?).

$$C_0 O_0 + C_1 O_1 = 144 - 143 \text{ MeV}.$$

The situation improves by lowering the scale ν ($97 - 66 \text{ MeV}$ for $\nu = 2 \text{ GeV}$ and $53 - 21 \text{ MeV}$ for $\nu = 1.5 \text{ GeV}$) and it also depends on the poorly known values of the condensates.

Situation $\Lambda_{\text{QCD}} \sim m\alpha_s^2$. Non-local condensate. Basically unknown. Formally **NNLO** effect ($\sim 50 \text{ MeV}$)

Error $\sim \pm 100 \text{ MeV}$.

Final results (RS= $N_m + \nu_f$)

RS scheme

$$m_{b,\text{RS}}(2 \text{ GeV}) = 4\,387_{-75}^{+75}(\text{US}/\text{N}^3\text{LO})_{+16}^{-16}(\alpha_s)_{+75}^{-73}(\text{RS}) \text{ MeV};$$

$$m_{b,\overline{\text{MS}}}(m_{b,\overline{\text{MS}}}) = 4\,203_{-67}^{+67}(\text{US}/\text{N}^3\text{LO})_{+27}^{-28}(\alpha_s)_{+16}^{-15}(\text{RS}) \text{ MeV}.$$

RS' scheme

$$m_{b,\text{RS}'}(2 \text{ GeV}) = 4\,782_{-75}^{+75}(\text{US}/\text{N}^3\text{LO})_{-12}^{+15}(\alpha_s)_{+31}^{-35}(\text{RS}') \text{ MeV};$$

$$m_{b,\overline{\text{MS}}}(m_{b,\overline{\text{MS}}}) = 4\,214_{-67}^{+67}(\text{US}/\text{N}^3\text{LO})_{+25}^{-25}(\alpha_s)_{+12}^{-15}(\text{RS}') \text{ MeV}.$$

We average the two values obtained for the $\overline{\text{MS}}$ mass. We then obtain (rounding)

$$m_{b,\overline{\text{MS}}}(m_{b,\overline{\text{MS}}}) = 4\,210_{-90}^{+90}(\text{theory})_{+25}^{-25}(\alpha_s) \text{ MeV},$$

The static singlet potential

Pineda (JPG)

The introduction of renormalons allows to obtain agreement between lattice simulations and perturbation theory.

$$E_s = 2m_{\text{OS}} + V_{s,\text{OS}} + \mathcal{O}(r^2)$$

$$E_s = 2m_{\text{RS}}(\nu_f) + V_{s,\text{RS}}(\nu_f) + \mathcal{O}(r^2)$$

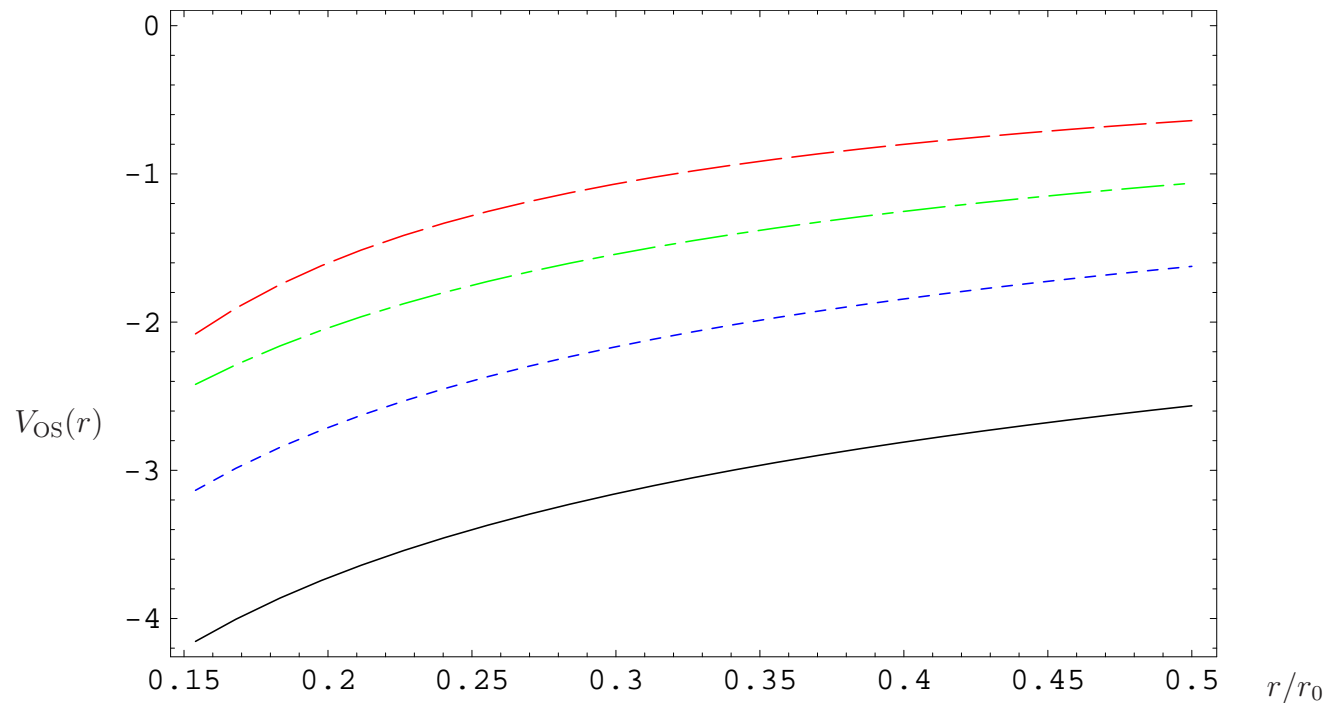


Figure 5: Plot of $V_{\text{OS}}(r)$ at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the leading single ultrasoft log (solid line). For the scale of $\alpha_s(\nu)$ $\nu = \text{constant}$. $\nu_{\text{us}} = 2.5 r_0^{-1}$.

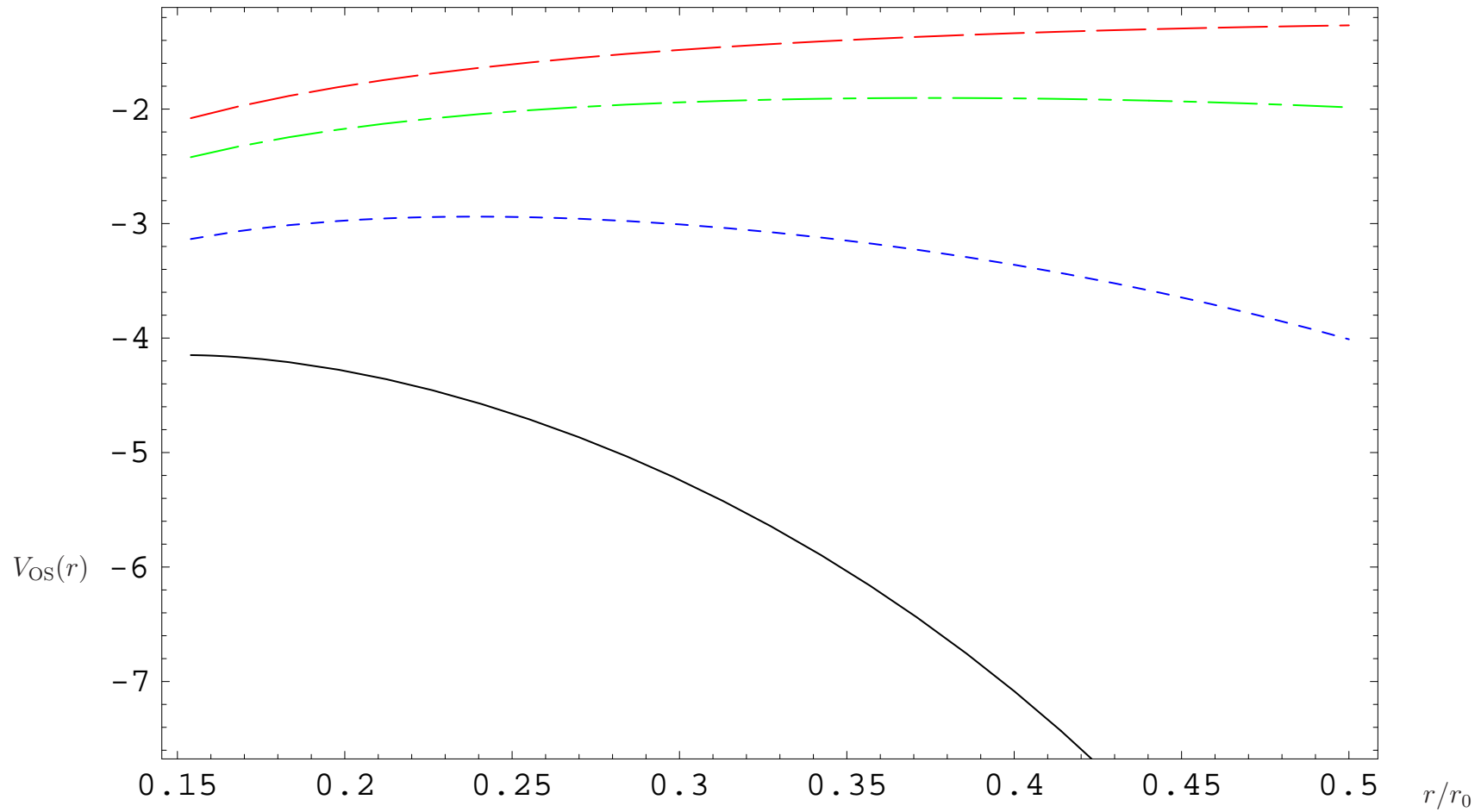


Figure 6: Plot of $V_{OS}(r)$ at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line). For the scale of $\alpha_s(\nu)$, we set $\nu = 1/r$. $\nu_{us} = 2.5 r_0^{-1}$.

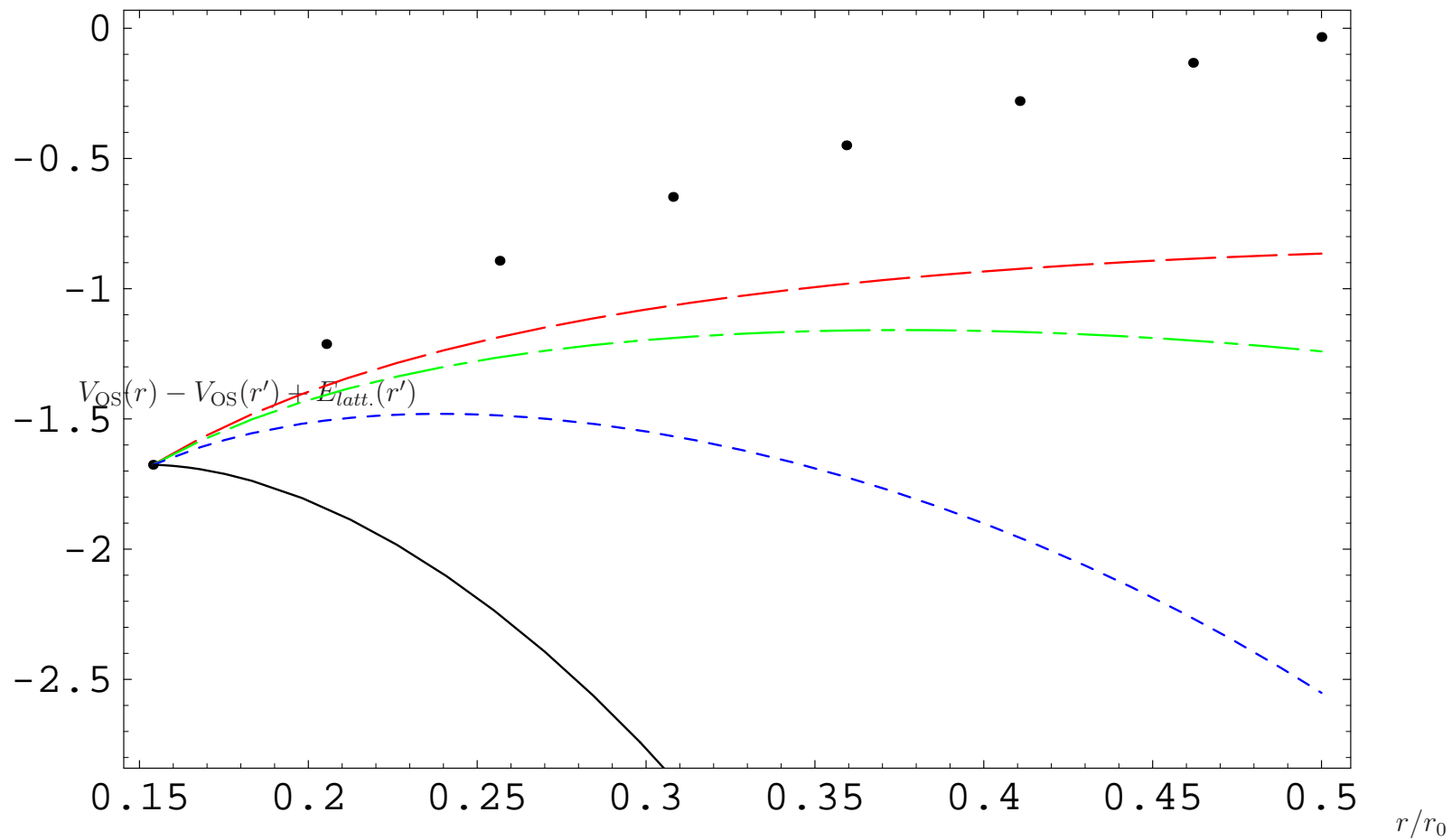


Figure 7: Plot of $V_{OS}(r) - V_{OS}(r') + E_{latt.}(r')$ versus r at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_s(\nu)$, we set $\nu = 1/r$. $\nu_{us} = 2.5 r_0^{-1}$.

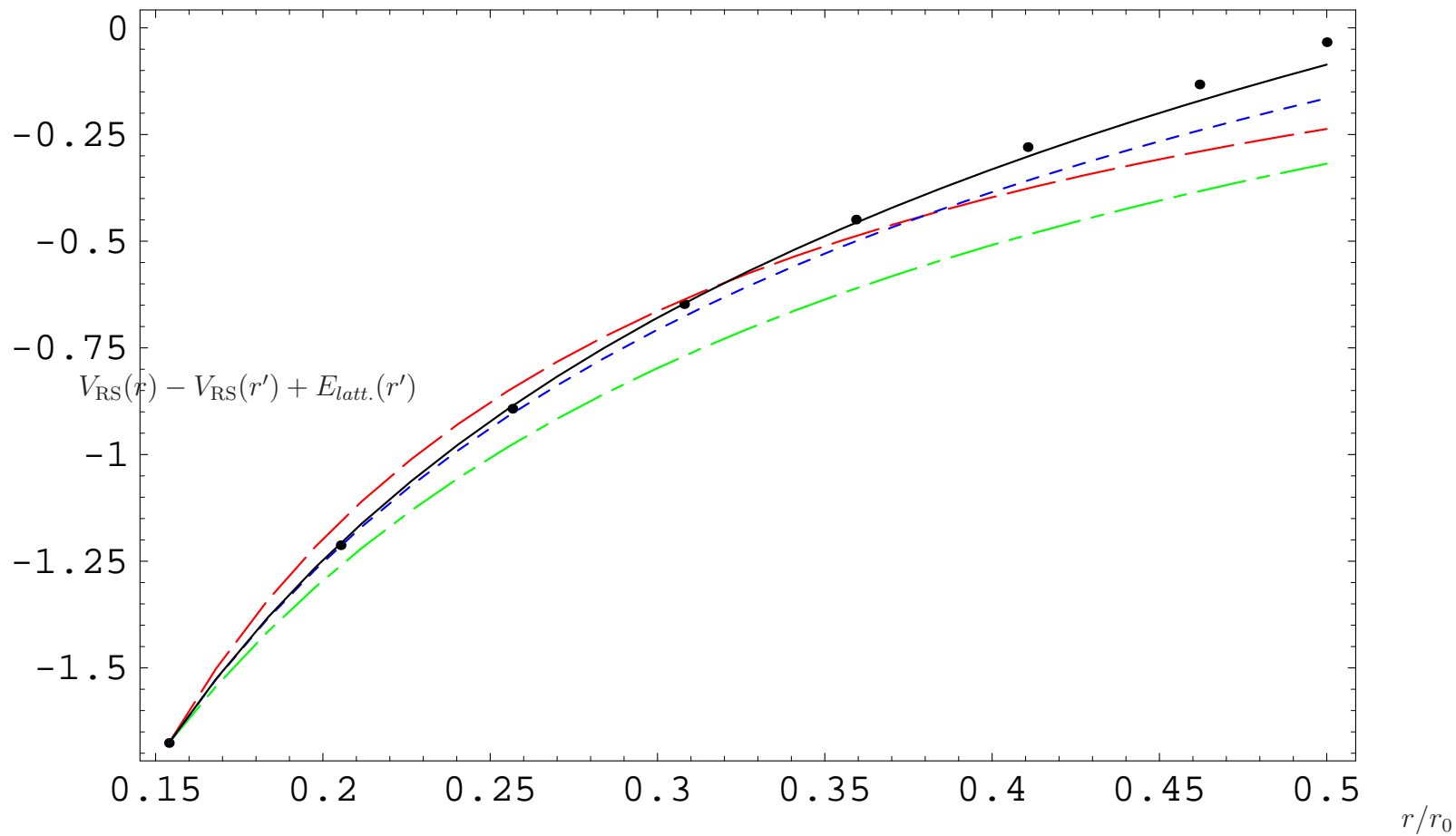


Figure 8: Plot of $V_{RS}(r) - V_{RS}(r') + E_{latt.}(r')$ versus r at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the leading single ultrasoft log (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_s(\nu)$ $\nu = \text{constant}$. $\nu_{us} = 2.5 r_0^{-1}$.

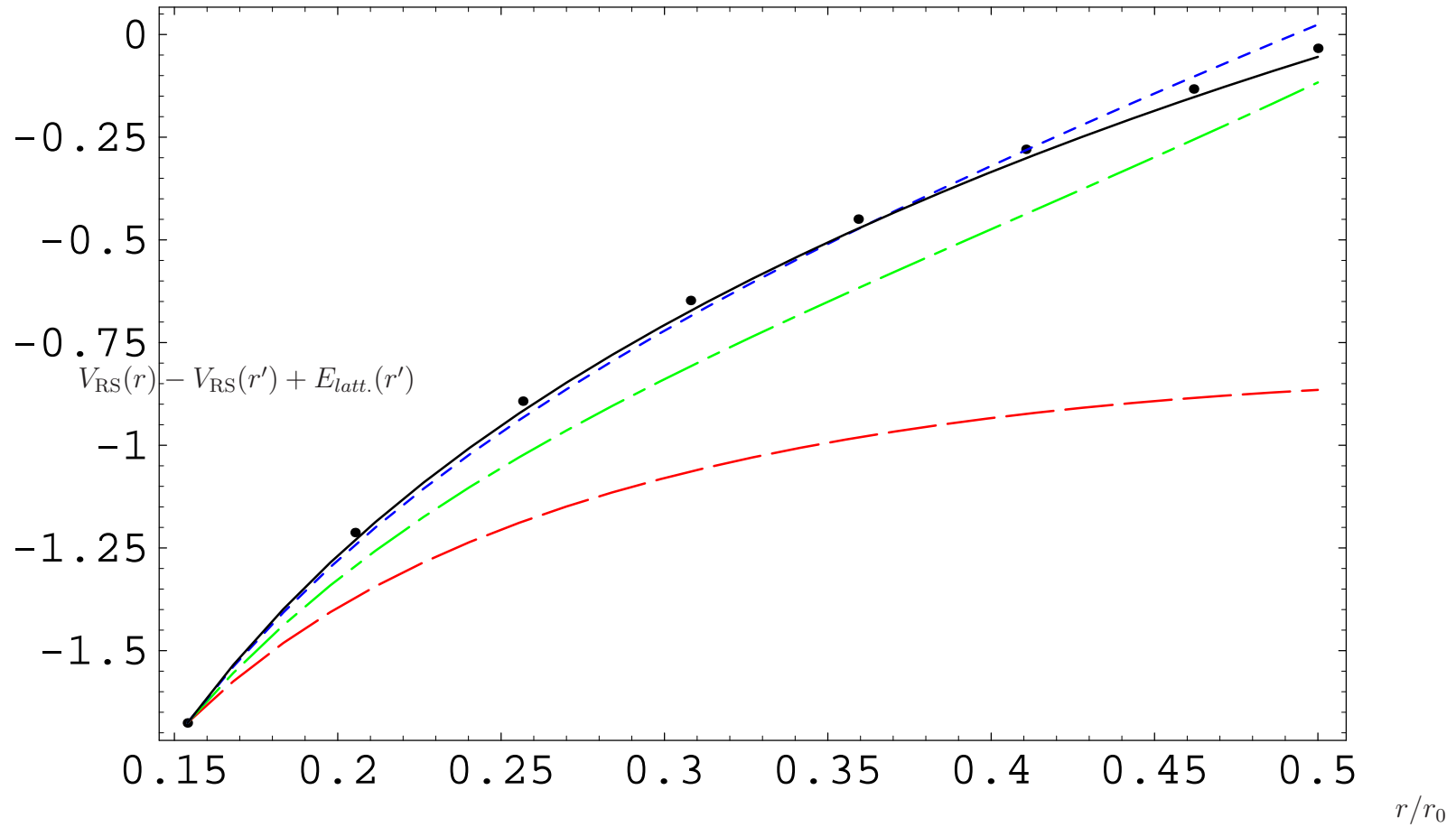


Figure 9: Plot of $V_{\text{RS}}(r) - V_{\text{RS}}(r') + E_{\text{latt.}}(r')$ versus r at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_s(\nu)$, we set $\nu = 1/r$. $\nu_f = \nu_{us} = 2.5 r_0^{-1}$.

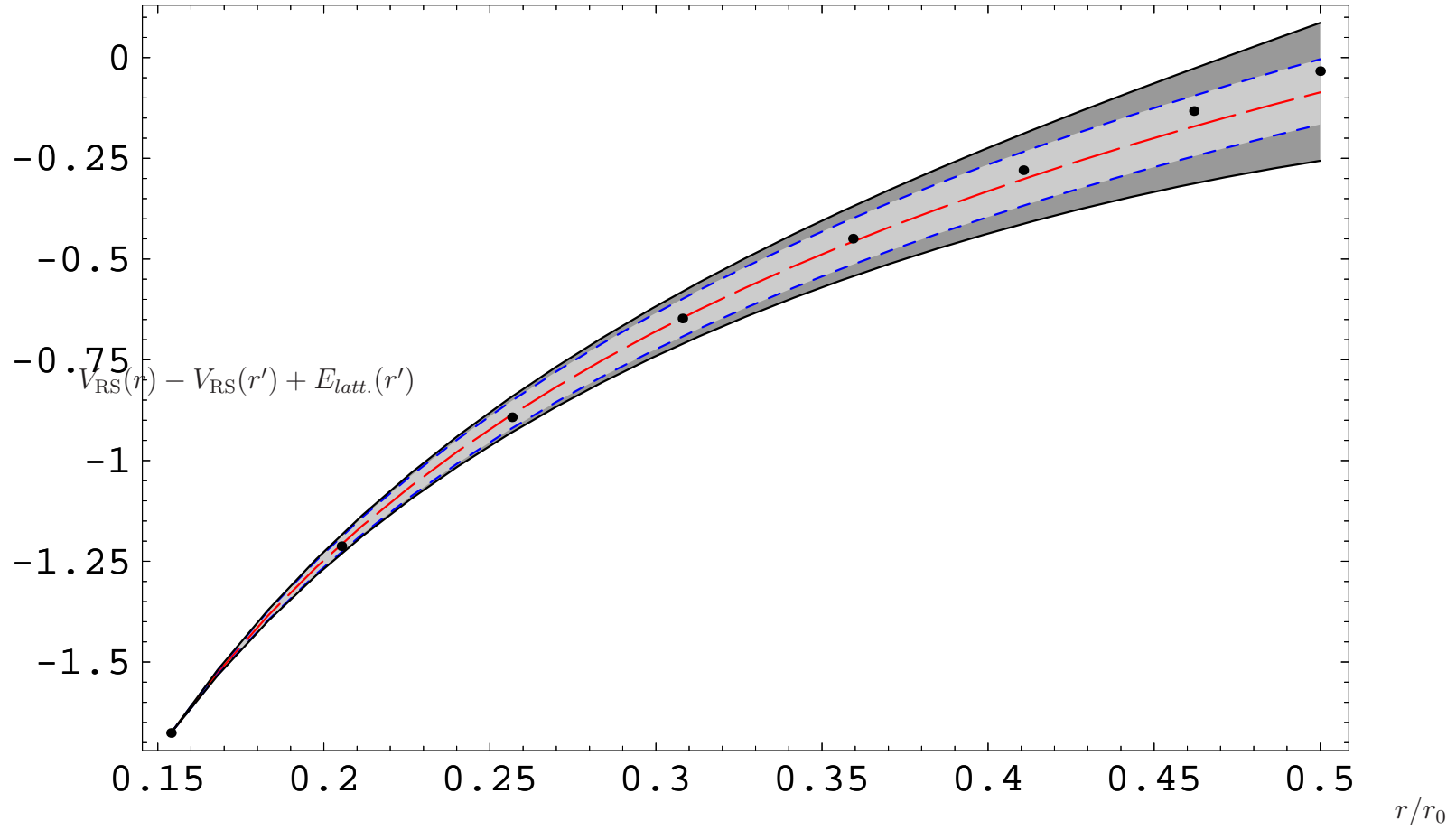


Figure 10: Plot of $V_{\text{RS}}(r) - V_{\text{RS}}(r') + E_{\text{latt.}}(r')$ versus r at tree (dashed line), one-loop (dash-dotted line), two-loops (dotted line) and three loops (estimate) plus the leading single ultrasoft log (solid line) compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_s(\nu)$ $\nu = \text{constant}$. $\nu_{\text{us}} = 2.5 r_0^{-1}$.

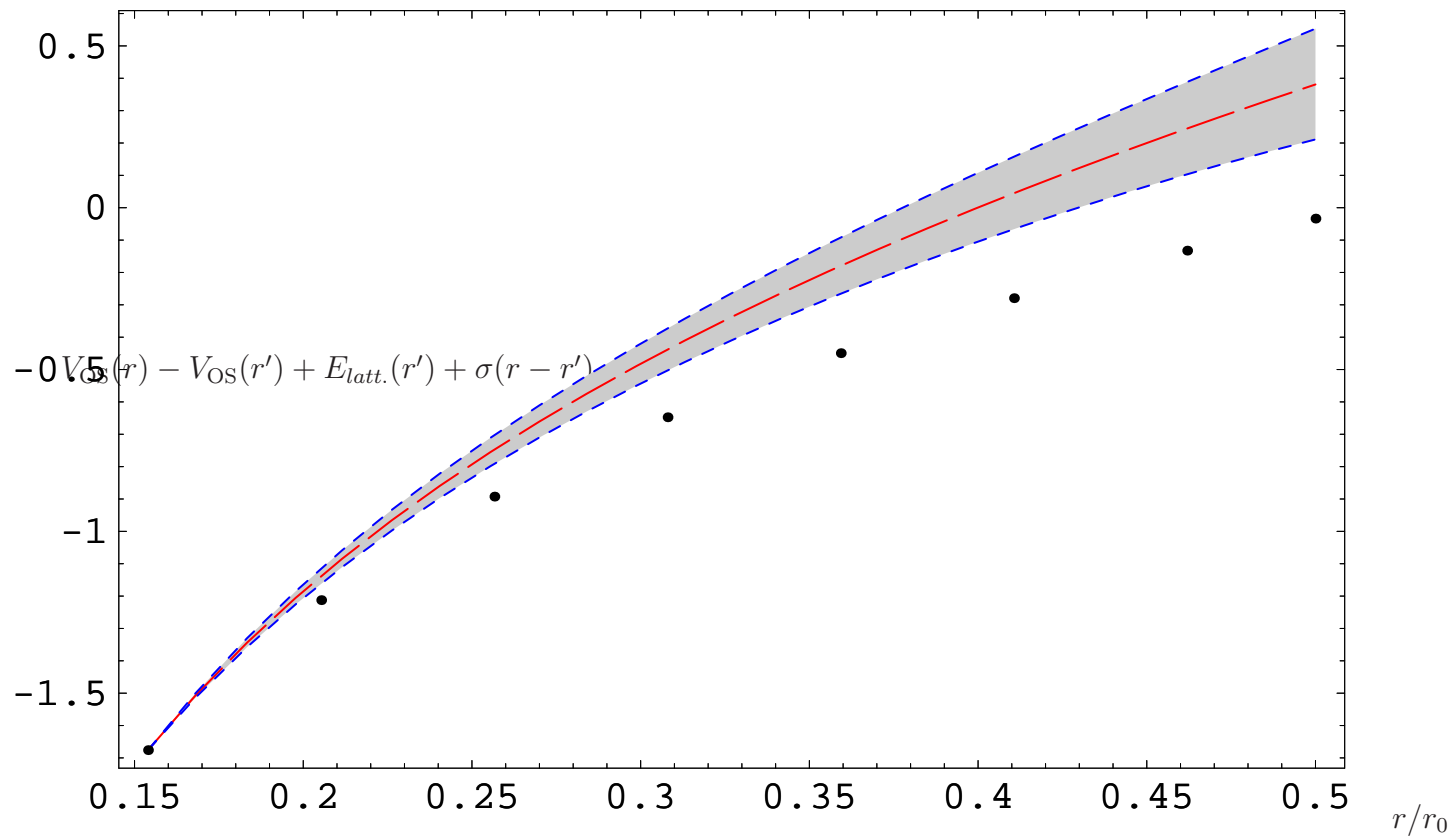


Figure 11: Plot of $V_{OS}(r) - V_{OS}(r') + E_{latt.}(r') + \sigma(r - r')$ versus r at three loops (estimate) with the leading ultrasoft log compared with the lattice simulations of Necco and Sommer. For the scale of $\alpha_s(\nu)$, we set $\nu = \text{constant}$. $\sigma = 1.35 r_0^{-2}$ and $\nu_{us} = 2.5 r_0^{-1}$.

Constraint on the size of nonperturbative effects for heavy quarkonium.
 No linear non-perturbative potential at short distances.

Static Hybrids versus gluelumps

$$E_H = 2m_{\text{OS}} + V_{o,\text{OS}} + \Lambda_H^{\text{OS}} + \mathcal{O}(r^2)$$

$$E_H = 2m_{\text{RS}}(\nu_f) + V_{o,\text{RS}}(\nu_f) + \Lambda_H^{\text{RS}}(\nu_f) + \mathcal{O}(r^2)$$

$$E_H = 2m_{\text{latt}}(1/a) + V_{o,\text{latt}}(1/a) + \Lambda_H^{\text{latt}}(1/a) + \mathcal{O}(r^2)$$

$$V_o = \frac{1}{2N_c} \frac{\alpha_s}{r} + \dots \text{ (two - loops : Kniehl, Penin, Schroeder, Smirnov, Steinhauser)}$$

$$2N_m + N_{V_o} + N_\Lambda = 0$$

Factorization scale $\nu_f \leftrightarrow 1/a$; $\delta m_{\text{RS}}(\nu_f)$, $\delta \Lambda_{\text{RS}}(\nu_f)$; $\delta m_{\text{latt}}(1/a)$, $\delta \Lambda_{\text{latt}}(1/a)$.

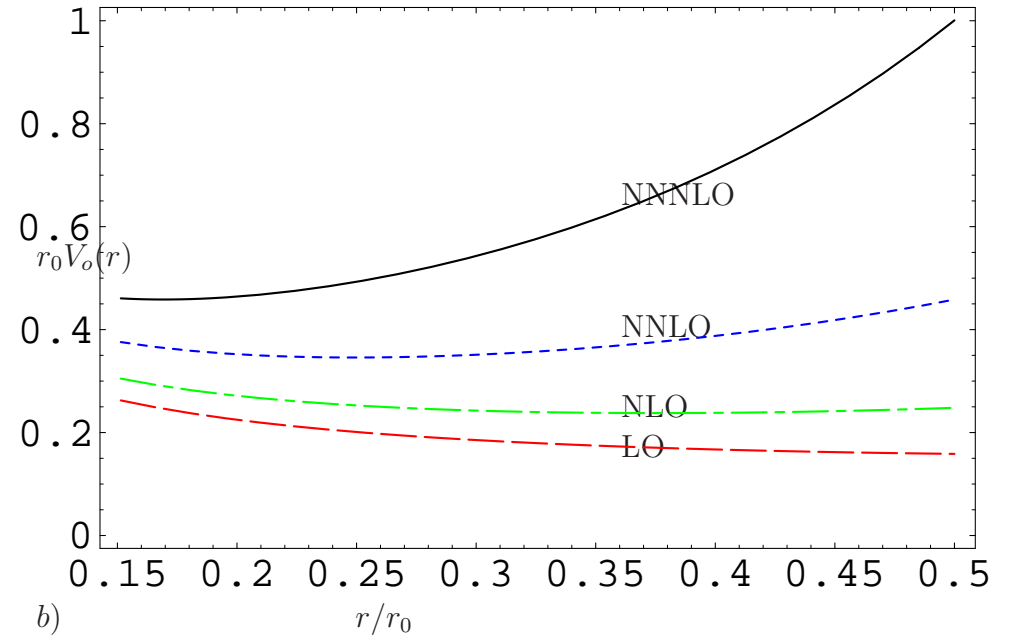
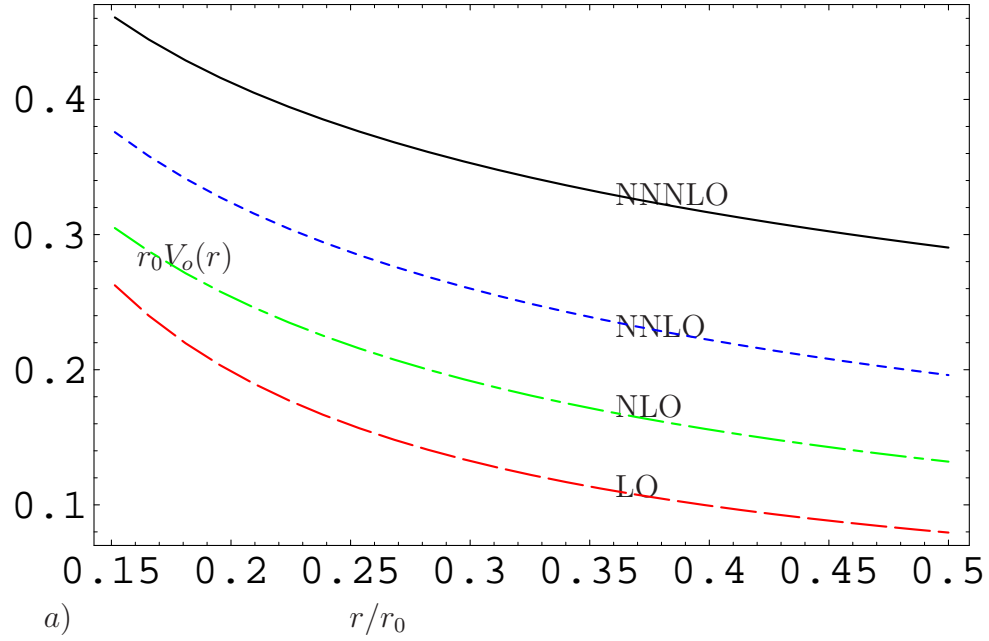


Figure 12: $r_0 V_o(r)$ (the octet potential in the OS scheme) at tree level (dashed lines), one-loop (dashed-dotted lines), two loops (dotted lines) and three loops (estimate) plus the leading single ultrasoft log (solid lines). Fig. a) corresponds to the scale $\nu = \nu_i$ and Fig. b) to $\nu = 1/r$. In both cases, $\nu_{us} = 2.5 r_0^{-1}$. Only the solid curves depend on this choice.

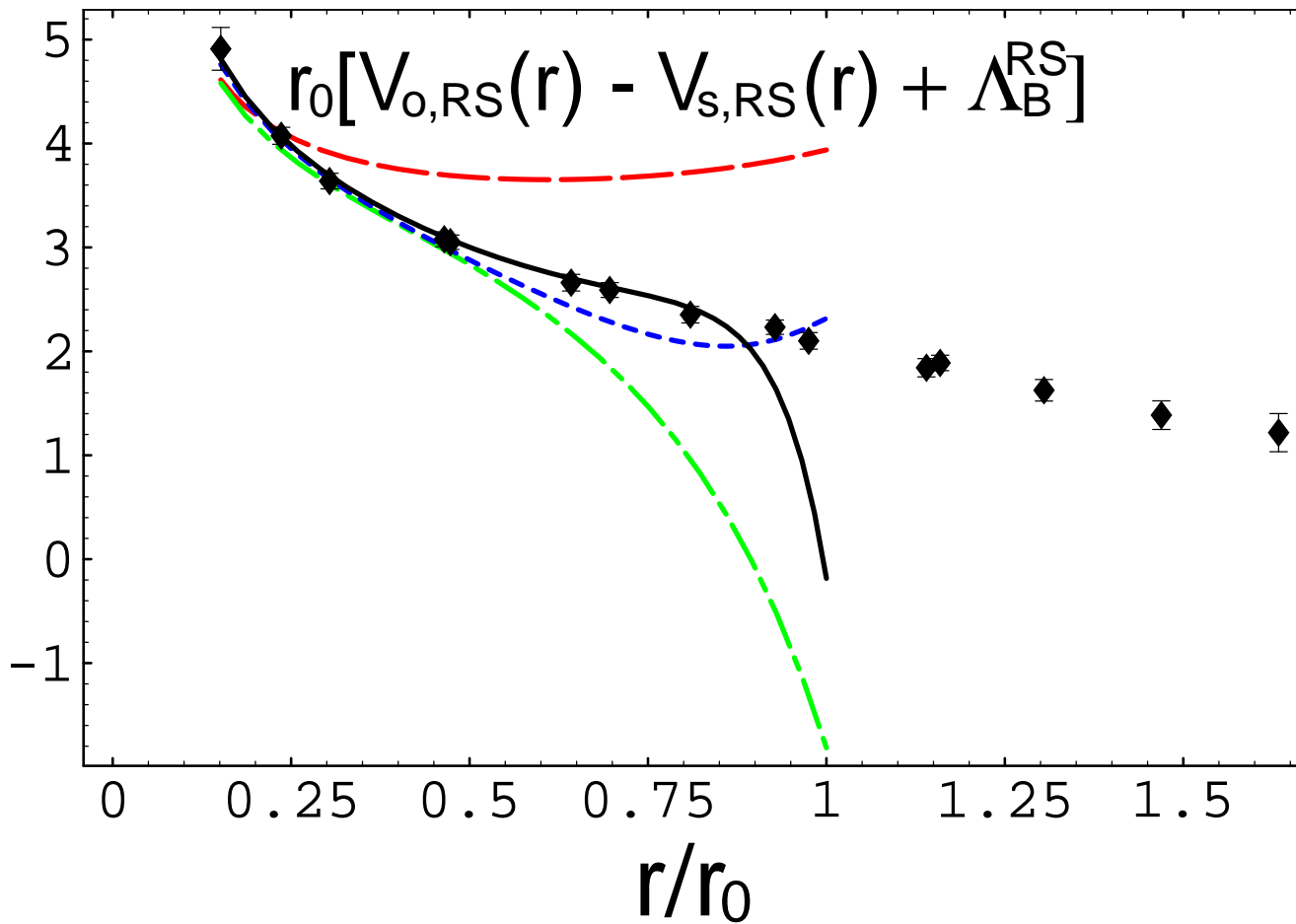


Figure 13: Splitting between the Π_u and the Σ_g^+ potentials and the comparison with the theoretical prediction with $\nu = 1/r$ for $\nu_f = 2.5 r_0^{-1}$. $r_0[(V_{0,RS} - V_{s,RS})(r) + \Lambda_B^{RS}]$ is plotted versus r at tree level (dashed line), one-loop (dashed-dotted line), two-loops (dotted line) and three loops (estimate) plus the RG expression for the ultrasoft logs (solid line). Bali-Pineda 2004.

Table 6: Absolute values for the gluelump masses in the continuum limit in the RS scheme at $\nu_f = 2.5 r_0^{-1} \approx 1$ GeV, in r_0 units and in GeV. Note that an additional uncertainty of about 10 % should be added to the last column to account for the quenched approximation. We also display examples of creation operators H for these states. The curly braces denote complete symmetrization of the indices. Bali-Pineda 2004.

J^{PC}	H	$\Lambda_H^{\text{RS}} r_0$	$\Lambda_H^{\text{RS}}/\text{GeV}$
1^{+-}	B_i	2.25(39)	0.87(15)
1^{--}	E_i	3.18(41)	1.25(16)
2^{--}	$D_{\{i}B_{j\}}$	3.69(42)	1.45(17)
2^{+-}	$D_{\{i}E_{j\}}$	4.72(48)	1.86(19)
3^{+-}	$D_{\{i}D_jB_k\}$	4.72(45)	1.86(18)
0^{++}	\mathbf{B}^2	5.02(46)	1.98(18)
4^{--}	$D_{\{i}D_jD_kB_l\}$	5.41(46)	2.13(18)
1^{-+}	$(\mathbf{B} \wedge \mathbf{E})_i$	5.45(51)	2.15(20)

Heavy-light systems

$$M_B = m_{\text{RS}}(\nu_f) + \bar{\Lambda}^{\text{RS}}(\nu_f) + \mathcal{O}(1/m)$$

$$M_B = m_{\text{latt}}(1/a) + \bar{\Lambda}^{\text{latt}}(1/a) + \mathcal{O}(1/m)$$

$$\bar{\Lambda}^{\text{RS}}(2.5 r_0^{-1}) = [1.17 \pm 0.08(\text{latt.}) \pm 0.13(\text{th.}) \pm 0.09(\Lambda_{\overline{\text{MS}}})] r_0^{-1} .$$

$$m_{b,\overline{\text{MS}}}(m_{b,\overline{\text{MS}}}) = [4191 \pm 29(\text{latt.}) \pm 47(\text{th.}) \pm 1(\Lambda_{\overline{\text{MS}}})] \text{ MeV} .$$

Perturbative running of $1/a$

$$2(\bar{\Lambda}_L(1/a) - \bar{\Lambda}_L(1/a')) = V_{s,L}(r; 1/a) - V_{s,L}(r; 1/a') = C_F \left(\frac{1}{a} - \frac{1}{a'} \right) v_1 \alpha_s + \dots .$$

$$\begin{aligned} \Lambda_H^L(1/a) - \Lambda_H^L(1/a') &= [V_{o,L}(r; 1/a) - V_{s,L}(r; 1/a)] - [V_{o,L}(r; 1/a') - V_{s,L}(r; 1/a')] \\ &= \frac{C_A}{2} \left(\frac{1}{a} - \frac{1}{a'} \right) v'_1 \alpha_s + \dots , \end{aligned}$$

$$\bar{\Lambda}_{\text{pot}}^L(a) = \frac{1}{2} V_s^L(r_0; a) + \Delta.$$

$$\left. \begin{array}{l} \bar{\Lambda}_L(1/a) \quad \iff \quad \bar{\Lambda}_L(1/a') \\ \Downarrow (\nu_f = 1/a) \quad \Downarrow (\nu'_f = 1/a') \\ \bar{\Lambda}_{\text{RS}}(\nu_f) \quad \iff \quad \bar{\Lambda}_{\text{RS}}(\nu'_f) \end{array} \right\} \text{The circle can be closed using perturbation theory}$$

and a similar circle applies for $V_s^L(r; 1/a)$, $V_o^L(r; 1/a)$ and $\Lambda_H^L(1/a)$.

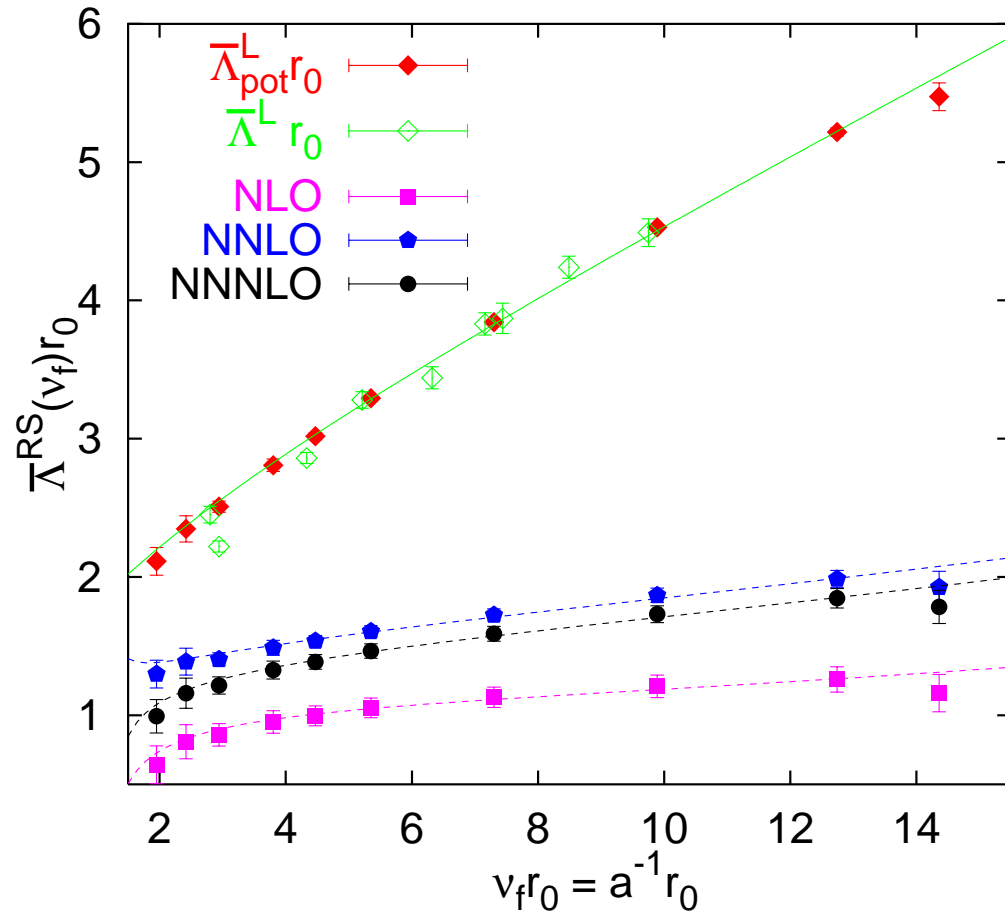


Figure 14: The binding energy $\bar{\Lambda}_{\text{pot}}^L$, in the lattice scheme (full diamonds), in comparison with $\bar{\Lambda}^L$ (open diamonds). The constant Δ has been adjusted by requiring agreement between the two data sets at $r_0 \approx 7.3 a$. The uncertainty of $\Delta = (0.988 \pm 0.067) r_0^{-1}$ is not included into the errors. NLO, NNLO and NNNLO refer to transformations of $\bar{\Lambda}_{\text{pot}}^L$ into the RS scheme to different orders in perturbation theory. The solid line corresponds to the NNNLO expectation with $\Lambda_{\overline{\text{MS}}} \approx 0.602 r_0^{-1}$, and the central value, $\bar{\Lambda}^{\text{RS}}(\nu_f = 9.76 r_0^{-1}) = 1.70 r_0^{-1}$. Bali-Pineda 2004.

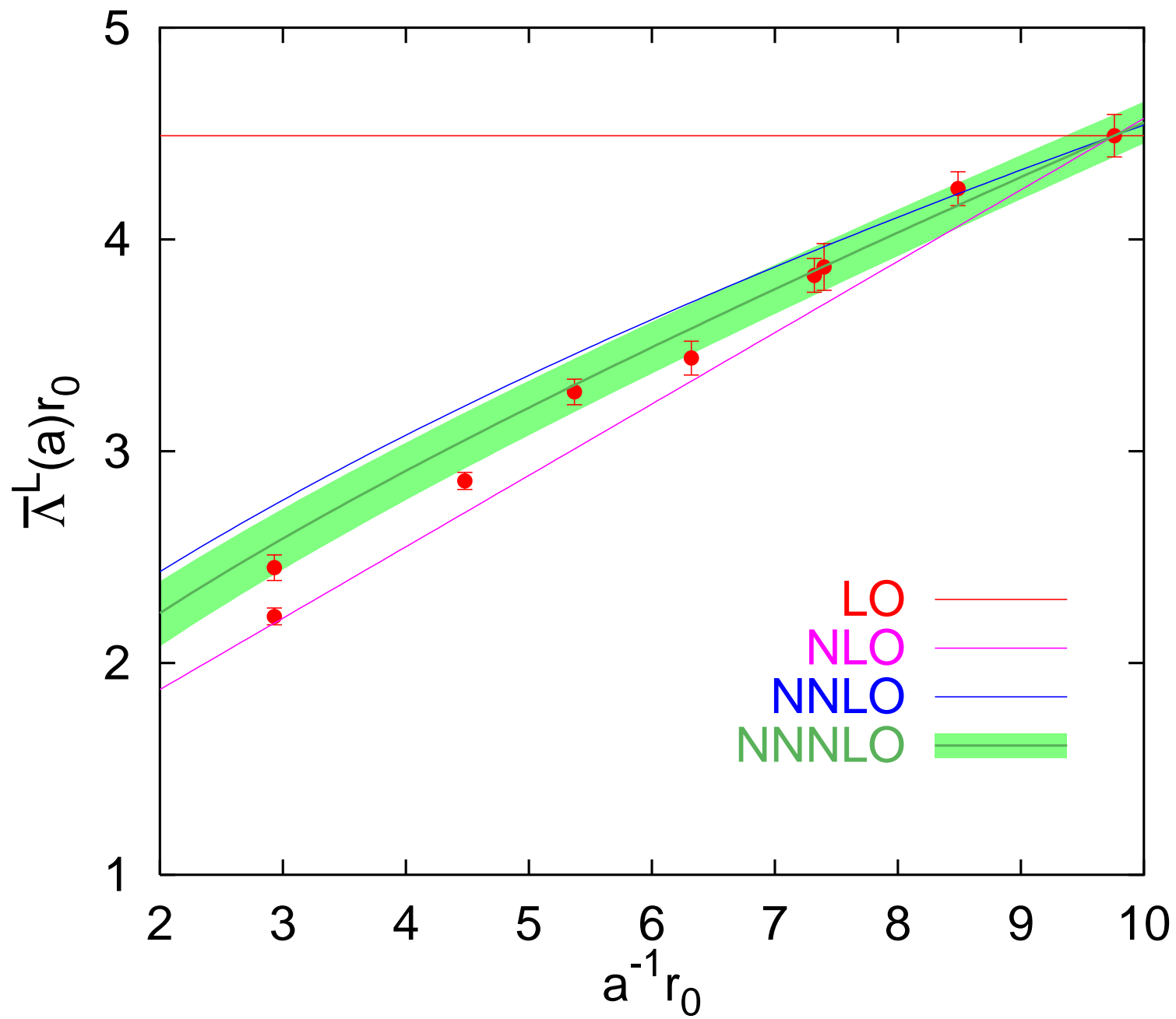


Figure 15: Perturbative running of the binding energy $\bar{\Lambda}$ in the lattice scheme, in comparison with lattice data, starting at the smallest available lattice spacing. The NNNLO error band incorporates the error due to the uncertainty in $\Lambda_{\overline{MS}}$, and the statistical error. Bali-Pineda(2004).

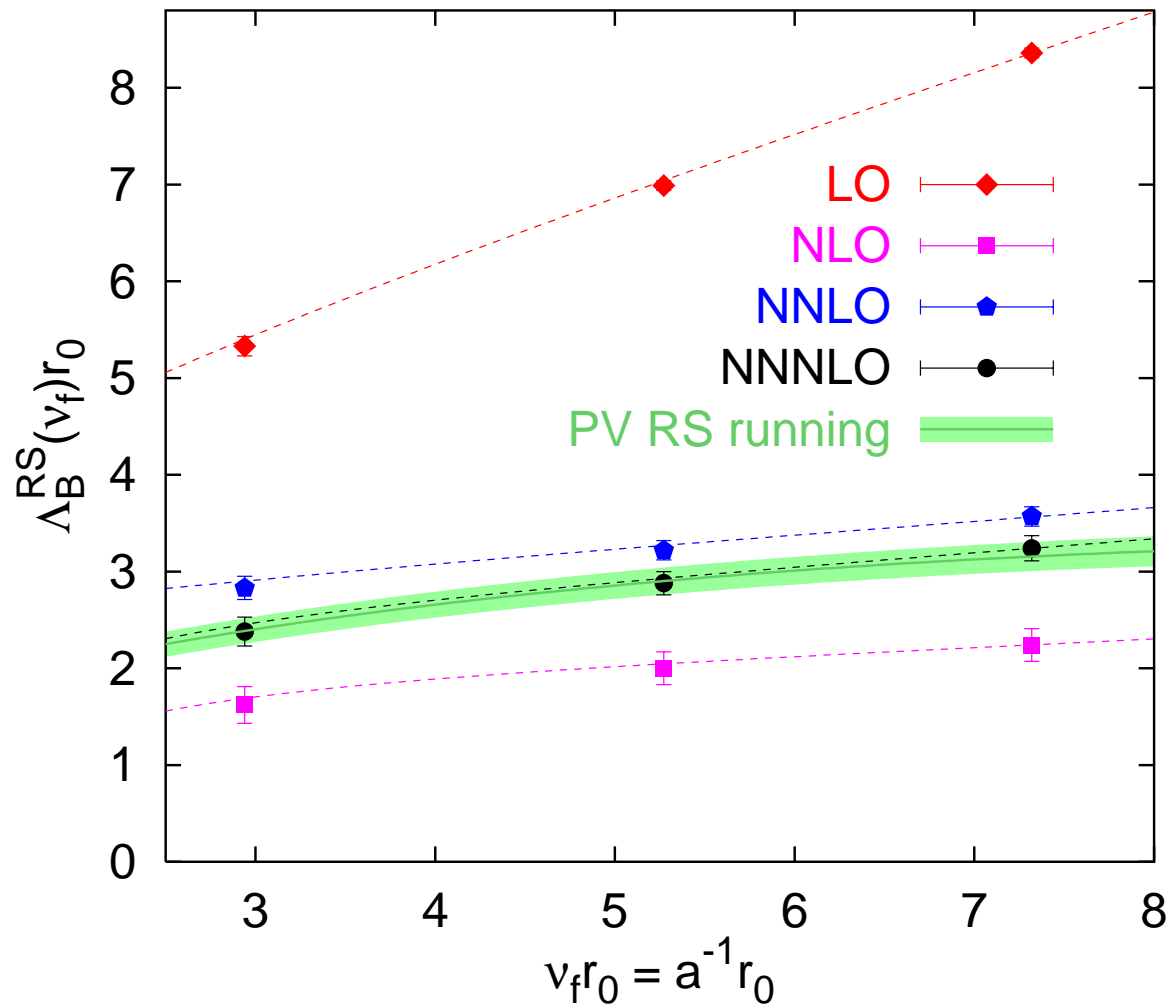


Figure 16: The lowest glueball mass Λ_B^L as obtained on the lattice (diamonds), as well as converted into the RS scheme at NLO (squares), NNLO (pentagons) and NNNLO* (NNNLO estimate, circles). The error band corresponds to the result for Λ_B^{RS} without the “theoretical” error, run to different scales, according to the PV prescription. Bali-Pineda 2004.

Conclusions

- Complete factorization of scales within dimensional regularization for nonrelativistic systems (by using effective field theories: **HQET**, **pNRQCD**, ...).
- Renormalon free working scheme preserving the power counting rules.

- Calculation of the **normalization constant** of the first renormalon of the pole mass and static potential.

$$N_m = 0.55 \quad N_{V_s} = -1.08 \quad (n_f = 4)$$

- Estimates of the **higher order coefficients** of the perturbative series relating the pole mass with the $\overline{\text{MS}}$ mass (and the singlet static potential with α_s) have been obtained without relying on the large β_0 approximation.
- Determination of the $\overline{\text{MS}}$ bottom and charm masses.

$$m_{b,\overline{\text{MS}}}(m_{b,\overline{\text{MS}}}) = 4 \, 210_{-90}^{+90}(\text{theory})_{+25}^{-25}(\alpha_s) \text{ MeV},$$

$$m_{c,\overline{\text{MS}}}(m_{c,\overline{\text{MS}}}) = 1 \, 210_{-70}^{+70}(\text{theory})_{-65}^{+65}(m_{b,\overline{\text{MS}}})_{+45}^{-45}(\lambda_1) \text{ MeV}.$$

- Good description of the (lattice) static singlet and octet potential at short distances with perturbation theory.
- Determination of the **gluelump** masses, $\bar{\Lambda}$ and m_b mass from lattice.