Parafermion vertex operator algebras

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1. Introduction

• Z-algebras (Lepowsky-Wilson, Lepowsky-Primc(1980s)): Study representations for affine Kac-Moody algebras.

• Parafermion conformal field theory (Zamolodchikov-Fateev(1985)): New conformal field theory related to an affine Kac-Moody algebra and its Heisenberg subalgebra.

• Coset construction (Goddard-Kent-Olive(1986)): The commutant of subVOA in a big VOA.

Parafermion vertex operator algebra: The commutant of the Heisenberg subVOA in affine VOA.

The goal of this talk: Study the structure and representation theory of parafermion VOA.

Let $k \geq 1$ be an integer,

$V(k,0) - --$ the level k generalized verma module for the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$, where $\mathfrak{g}$ is a finite dimensional simple Lie algebra.

$N(\mathfrak{g},k) - --$ the commutant of the Heisenberg subVOA in the VOA $V(k,0)$.

$\mathcal{L}(k,0) - --$ the level k irreducible highest weight module for the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$.

$K(\mathfrak{g},k) - --$ the commutant of the Heisenberg subVOA in the VOA $\mathcal{L}(k,0)$. $K(\mathfrak{g},k)$ is called the parafermion VOA.

Remark: $\mathcal{L}(k,0)$ is the simple quotient of $V(k,0)$ and $K(\mathfrak{g},k)$ is the simple quotient of $N(\mathfrak{g},k)$.

Conjectures (Dong-Lam-Yamada):

• the generators of $N(sl_2, k)$ and $K(sl_2, k)$.

• $K(sl_2, k)$ is rational and C_2-cofinite.

If $k \leq 6$, the conjectures have been solved by Dong-Lam-Yamada.

For general k, the generator conjecture for $N(sl_2, k)$ and $K(sl_2, k)$ has been solved by Dong-Lam-Wang-Yamada.

The main result:

• Generalize the generator result from $\widehat{sl_2}$ to any affine Kac-Moody algebra $\hat{\mathfrak{g}}$. Specifically, the parafermion VOA $K(\mathfrak{g}, k)$ is finitely generated.

• Obtain the C_2-cofiniteness for general parafermion vertex operator algebra $K(\mathfrak{g}, k)$ from the regularity of parafermion VOA $K(sl_2, k)$ for all k.

2. VOA $V(k,0)$ and $V(k,0)(0)$

$\mathfrak{g}$ $-$ $--$ a finite dimensional simple Lie algebra.

$\mathfrak{h}$ $-$ $--$ Cartan subalgebra of $\mathfrak{g}$.

Δ $-$ $--$ root system. Q $-$ $--$root lattice.

$\langle,\rangle$ $-$ $--$ an invariant symmetric nondegenerate bilinear form on $\mathfrak{g}$ such that $\langle\alpha,\alpha\rangle = 2$ if α is a long root.

$\mathfrak{h} \mapsto \mathfrak{h}^*$: $\alpha(h) = \langle t_\alpha, h\rangle$ for any $h \in \mathfrak{h}$.

$\{\alpha_1, \cdots, \alpha_l\}$ $-$ $--$ simple roots. θ $-$ $--$ highest root.

$\mathfrak{g}_\alpha$ $-$ $--$ root space associated to the root $\alpha \in \Delta$.

For $\alpha \in \Delta_+$, we fix $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ and $h_\alpha = \frac{2}{\langle\alpha,\alpha\rangle}t_\alpha \in \mathfrak{h}$ such that $[x_\alpha, x_{-\alpha}] = h_\alpha, [h_\alpha, x_{\pm\alpha}] = \pm 2x_{\pm\alpha}$. That is, $\mathfrak{g}^\alpha = \mathbb{C}x_\alpha + \mathbb{C}h_\alpha + \mathbb{C}x_{-\alpha}$ is isomorphic to sl_2.

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be the affine Lie algebra with bracket:

$$[a(m), b(n)] = [a, b](m + n) + m\langle a, b\rangle\delta_{m+n,0}k,$$

where $a(n) = a \otimes t^n$. Let $k \geq 1$ be an integer and

$$V(k, 0) = V_{\widehat{\mathfrak{g}}}(k, 0) = \mathrm{Ind}_{\mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}K}^{\widehat{\mathfrak{g}}}\mathbb{C}$$

the induced $\widehat{\mathfrak{g}}$-module such that $\mathfrak{g} \otimes \mathbb{C}[t]$ acts as 0 and K acts as k on $\mathbf{1} = 1$.

- $V(k,0)$ is a VOA with vacuum vector $\mathbf{1}$.

  Virasoro vector:

$$\omega_{\text{aff}} = \frac{1}{2(k+h^\vee)}\Big(\sum_{i=1}^{l} h_i(-1)h_i(-1)\mathbf{1} + \sum_{\alpha\in\Delta}\frac{\langle\alpha,\alpha\rangle}{\langle\theta,\theta\rangle}x_\alpha(-1)x_{-\alpha}(-1)\mathbf{1}\Big)$$

  Central charge: $\frac{k\dim\mathfrak{g}}{k+h^\vee}$.

Where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and $\{h_i|i=1,\cdots,l\}$ is an orthonormal basis of $\mathfrak{h}$.

For $\lambda \in \mathfrak{h}^*$, set

$$V(k,0)(\lambda) = \{v \in V(k,0) | h(0)v = \lambda(h)v, \forall\, h \in \mathfrak{h}\}.$$

Then we have

$$V(k,0) = \oplus_{\lambda \in Q} V(k,0)(\lambda).$$

We see that $V(k,0)(0)$ is a subVOA of $V(k,0)$ with the same Virasoro vector ω_{aff} and each $V(k,0)(\lambda)$ is a module for $V(k,0)(0)$.

Proposition 2.1. *VOA $V(k,0)(0)$ is generated by vectors $\alpha_i(-1)\mathbf{1}$ and $x_{-\alpha}(-2)x_\alpha(-1)\mathbf{1}$ for $1 \le i \le l, \alpha \in \Delta_+$.*

3. VOA $N(\mathfrak{g}, k)$

$M(k)$ — — — Heisenberg subVOA of $V(k,0)$ generated by $h(-1)\mathbf{1}$ for $h \in \mathfrak{h}$.

Virasoro element: $\omega_{\mathfrak{h}} = \frac{1}{2k}\sum_{i=1}^{l} h_i(-1)h_i(-1)\mathbf{1}$.

Central charge: l.

Note that $V(k,0)$ and $V(k,0)(\lambda)$, $\lambda \in Q$ are completely reducible $M(k)$-modules. That is,

$$V(k,0) = \oplus_{\lambda\in Q} M_{\widehat{\mathfrak{h}}}(k,\lambda) \otimes N_\lambda,$$

$$V(k,0)(\lambda) = M_{\widehat{\mathfrak{h}}}(k,\lambda) \otimes N_\lambda$$

where $M_{\widehat{\mathfrak{h}}}(k,\lambda)$ denotes an irreducible highest weight module for $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$ with a highest weight vector v_λ such that $h(0)v_\lambda = \lambda v_\lambda$ and

$$N_\lambda = \{v \in V(k,0) \mid h(m)v = \lambda(h)\delta_{m,0}v \text{ for } h \in \mathfrak{h}, m \geq 0\}$$

is the space of highest weight vectors with highest weight λ for $\widehat{\mathfrak{h}}$.

Note that $N(\mathfrak{g},k)=N_0$ is the commutant of $M(k)$ in $V(k,0)$. The commutant $N(\mathfrak{g},k)$ is a VOA with the Virasoro vector $\omega=\omega_{\text{aff}}-\omega_{\mathfrak{h}}$ whose central charge is $\frac{k\dim\mathfrak{g}}{k+h^\vee}-l$.

If $\alpha\in\triangle_+$, we let

$$\omega_\alpha=\frac{1}{2k_\alpha(k_\alpha+2)}(-k_\alpha h_\alpha(-2)\mathbf{1}-h_\alpha(-1)^2\mathbf{1}+2k_\alpha x_\alpha(-1)x_{-\alpha}(-1)\mathbf{1}),$$

$$\begin{aligned}W_\alpha^3&=k_\alpha^2h_\alpha(-3)\mathbf{1}+3k_\alpha h_\alpha(-2)h_\alpha(-1)\mathbf{1}+2h_\alpha(-1)^3\mathbf{1}\\&-6k_\alpha h_\alpha(-1)x_\alpha(-1)x_{-\alpha}(-1)\mathbf{1}+3k_\alpha^2x_\alpha(-2)x_{-\alpha}(-1)\mathbf{1}\\&-3k_\alpha^2x_\alpha(-1)x_{-\alpha}(-2)\mathbf{1},\end{aligned}$$

where $k_\alpha=\frac{\langle\theta,\theta\rangle}{\langle\alpha,\alpha\rangle}k$.

Remark. For $\alpha\in\Delta$, $\omega_\alpha=\omega_{-\alpha}$, $W_{-\alpha}^3=-W_\alpha^3$, and $\omega=\sum_{\alpha\in\Delta_+}\frac{k(k_\alpha+2)}{k_\alpha(k+h^\vee)}\omega_\alpha$.

Let $\widehat{P}_\alpha$ be the subVOA of $N(\mathfrak{g}, k)$ generated by ω_α and W_α^3. Then $\widehat{P}_\alpha$ is isomorphic to $N(sl_2, k_\alpha)$.

Theorem 3.1.(Dong-Wang) *VOA $N(\mathfrak{g}, k)$ is generated by $\dim \mathfrak{g} - l$ vectors ω_α and W_α^3 for $\alpha \in \Delta_+$. That is, $N(\mathfrak{g}, k)$ is generated by subalgebras $\widehat{P}_\alpha$ for $\alpha \in \Delta_+$.*

Remark 3.2. *VOA $N(sl_2, k)$ is isomorphic to $W(2, 3, 4, 5)$ with primary fields of conformal weight 3,4,5*(Dong-Lam-Wang-Yamada).

Remark 3.3. *VOA $N(\mathfrak{g}, k)$ and its quotient $K(\mathfrak{g}, k)$ are of moonshine type. That is, their weight zero subspaces are 1-dimensional and weight one subspaces are zero.*

4. Parafermion VOA $K(\mathfrak{g}, k)$

$\mathcal{J}$ — — a unique maximal ideal of $V(k,0)$ which generated by a weight $k+1$ vector $x_\theta(-1)^{k+1}\mathbf{1}$ [K], where θ is the highest root of $\mathfrak{g}$.

$\mathcal{L}(k,0) = V(k,0)/\mathcal{J}$ — — — a simple, rational VOA associated to affine Lie algebra $\widehat{\mathfrak{g}}$.

Remark. Heisenberg VOA $M(k)$ is again a simple subVOA of $\mathcal{L}(k,0)$ and $\mathcal{L}(k,0)$ is a completely reducible $M(k)$-module. We have a decomposition

$$\mathcal{L}(k,0) = \oplus_{\lambda\in Q} M_{\widehat{\mathfrak{h}}}(k,\lambda) \otimes K_\lambda$$

as modules for $M(k)$, where

$$K_\lambda = \{v \in \mathcal{L}(k,0) \,|\, h(m)v = \lambda(h)\delta_{m,0}v \text{ for } h \in \mathfrak{h}, m \geq 0\}.$$

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Set $K(\mathfrak{g},k)=K_0$. Then $K(\mathfrak{g},k)$ is the commutant of $M(k)$ in $\mathcal{L}(k,0)$ and is called the parafermion VOA associated to the irreducible highest weight module $\mathcal{L}(k,0)$ for $\widehat{\mathfrak{g}}$. $K(\mathfrak{g},k)$ are conjectured to be rational, C_2-cofinite VOA.

Similarly, $\mathcal{J}$ is completely reducible as a $M(k)$-module. Hence

$$\mathcal{J}=\oplus_{\lambda\in Q}M_{\widehat{\mathfrak{h}}}(k,\lambda)\otimes(\mathcal{J}\cap N_\lambda).$$

In particular, $\widetilde{\mathcal{I}}=\mathcal{J}\cap N(\mathfrak{g},k)$ is an ideal of $N(\mathfrak{g},k)$ and $K(\mathfrak{g},k)\cong N(\mathfrak{g},k)/\widetilde{\mathcal{I}}$. We proved that $\widetilde{\mathcal{I}}$ is the unique maximal ideal of $N(\mathfrak{g},k)$. Thus $K(\mathfrak{g},k)$ is a simple VOA.

We still use ω_{aff}, $\omega_{\mathfrak{h}}$, ω_α, W_α^3 to denote their images in $\mathcal{L}(k,0)=V(k,0)/\mathcal{J}$.

The following result is a direct consequence of Theorem 3.1.

Theorem 4.1.(Dong-Wang) *The simple VOA* $K(\mathfrak{g}, k)$ *is generated by* ω_α, W_α^3 *for* $\alpha \in \Delta_+$.

Next, we study the ideal $\widetilde{\mathcal{I}}$ of $N(\mathfrak{g}, k)$.

Proposition 4.2. *The maximal ideal* $\widetilde{\mathcal{I}}$ *of* $N(\mathfrak{g}, k)$ *is generated by* $x_{-\theta}(0)^{k+1} x_\theta(-1)^{k+1}\mathbf{1}$.

For $\alpha \in \Delta_+$, we let P_α be the subVOA of $K(\mathfrak{g}, k)$ generated by ω_α and W^3_α. Then P_α is a quotient of $\widehat{P}_\alpha$. A natural question is whether or not P_α is a simple VOA.

Proposition 4.3. *Let $\alpha \in \Delta_+$. Then the subVOA P_α of $K(\mathfrak{g}, k)$ is a simple VOA isomorphic to the parafermion VOA $K(sl_2, k_\alpha)$ associated to sl_2.*

Remark 4.4. *We expect from Proposition 4.3 that the role of $K(sl_2, k_\alpha)$ played in the theory of parafermion VOA is similar to the role of sl_2 played in the theory of Kac-Moody Lie algebras. So a study of structural and representation theory for $K(sl_2, k_\alpha)$ becomes extremely important for general parafermion VOA.*

5. C_2-cofiniteness of $K(\mathfrak{g}, k)$

For a VOA V,

- V is rational: the category of admissible modules is semisimple, that is, every admissible module is a direct sum of simple admissible modules.

- V is C_2-cofinite: the subspace $C_2(V)$ spanned by $u_{-2}v$ for all $u, v \in V$ has finite codimension in V.

- V is regular: every weak module is a direct sum of simple ordinary modules.

- Conjecture(Dong-Li-Mason): rationality $\Leftrightarrow$ regularity, rationality $\Rightarrow$ C_2-cofiniteness.
- regularity $\Rightarrow$ rationality(by definition).
- regularity $\Rightarrow$ C_2-cofiniteness(Li).
- rationality$+C_2$-cofiniteness $\Rightarrow$ regularity(Abe-Buhl-Dong).

Theorem 5.1.(Dong-Lam-Yamada) *$K(sl_2,k)$ is a simple rational and C_2-cofinite for $k \le 6$. That is, $K(sl_2,k)$ is regular for such k.*

The main theorem of this section is

Theorem 5.2.(Dong-Wang) *If $K(sl_2,k)$ is rational and C_2-cofinite for all k, then $K(\mathfrak{g},k)$ is C_2-cofinite.*

Conjecture. *If $K(sl_2,k)$ is rational and C_2-cofinite for all k, then $K(\mathfrak{g},k)$ is rational.*

Theorem 5.3.(Dong-Wang) *$K(\mathfrak{g},k)$ is C_2-cofinite if $\mathfrak{g}$ is ADE type with $k \le 6$, $\mathfrak{g}$ is type G_2 with $k \le 2$, and $\mathfrak{g}$ is other type with $k \le 3$.*

6. The structure of $K(\mathfrak{g}, 1)$

Let $L(c, h)$ be the irreducible highest weight module for the Virasoro algebra with central charge c and highest weight h.

Fact. • Virasoro VOA $L(c, 0)$ is rational $\Leftrightarrow$ $c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$, where $p, q \in \{2, 3, 4 \ldots\}$ and $(p, q) = 1$.

• $\{L(c_{p,q}, h^{p,q}_{m,n}) | 0 < m < p, 0 < n < q\}$ is the set of all irreducible modules for the VOA $L(c_{p,q}, 0)$, where $h^{p,q}_{m,n} = \frac{(np-mq)^2-(p-q)^2}{4pq}$.

• $L(c_{p,q}, 0)$ is unitary as a Virasoro module $\Leftrightarrow$ $p = q + 1$.

• If $p = 3, q = 4$, $L(c_{p,q}, 0) = L(\frac{1}{2}, 0)$ is a rational VOA with 3 irreducible modules $L(\frac{1}{2}, h)$ with $h = 0, \frac{1}{2}, \frac{1}{16}$.

• If $p = 4, q = 5$, $L(c_{p,q}, 0) = L(\frac{7}{10}, 0)$ is a rational VOA with irreducible modules $L(\frac{7}{10}, h)$ with $h = 0, \frac{1}{2}, \frac{7}{16}, \frac{3}{80}, \frac{3}{2}, \frac{3}{5}, \frac{1}{10}$.

• If $p = 5, q = 6$, $L(c_{p,q}, 0) = L(\frac{4}{5}, 0)$ is a rational VOA with irreducible modules $\cdots$.

Fact. *$K(sl_2, 1) = \mathbb{C}$ and $K(sl_2, 2) \cong L(\frac{1}{2}, 0)$. In particular, $W_\alpha^3 = 0$ for $k = 1, 2$, where α is a root of sl_2.*

- If $\mathfrak{g}$ is of ADE types, then $K(\mathfrak{g}, 1) = \mathbb{C}$.

Non-simply laced simple Lie algebra:

• Recall that the subalgebra P_α of $K(\mathfrak{g}, 1)$ generated by $\omega_\alpha, W_\alpha^3$ with $\alpha \in \Delta_+$ is isomorphic to $K(sl_2, k_\alpha)$. That is,

$$P_\alpha \cong K(sl_2, 2) \quad \text{if} \ \langle \alpha, \alpha \rangle = 1.$$

$$P_\alpha \cong K(sl_2, 3) \quad \text{if} \ \langle \alpha, \alpha \rangle = \frac{2}{3}.$$

This implies

• If $\mathfrak{g}$ is of type B_l, C_l, F_4, $K(\mathfrak{g}, 1)$ is generated by ω_α with $\alpha \in \Delta_+$ being short roots.

• If $\mathfrak{g}$ is of type G_2, the situation is more complicated.

Result .

- $K(B_l, 1) = L(\frac{1}{2}, 0)$.
- $K(C_l, 1) \cong K(A_{l-1}, 2)$.
- $K(F_4, 1) \cong K(A_2, 2) = L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2})$.
- $K(G_2, 1) \cong K(A_1, 3) = L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$.

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Thank you!